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VISCOSITY SOLUTIONS OF HAMILTON-JACOBI EQUATIONS

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ABSTRACT

Problems involving Hamilton-Jacobi equations - which we take to be either of the stationary form $H(x,u,Du) = 0$ or of the evolution form $u_t + H(x,t,u,Du) = 0$, where Du is the spatial gradient of u - arise in many contexts. Classical analysis of associated problems under boundary and/or initial conditions by the method of characteristics is limited to local considerations owing to the crossing of characteristics. Global analysis of these problems has been hindered by the lack of an appropriate notion of solution for which one has the desired existence and uniqueness properties. In this work a notion of solution is proposed which allows, for example, solutions to be nowhere differentiable but for which strong uniqueness theorems, stability theorems and general existence theorems, as discussed herein, are all valid.

AMS(MOS) Subject Classification: 35F20, 35F25, 35L60

Key Words: Hamilton-Jacobi equations, uniqueness criteria

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VISCOSITY SOLUTIONS OF HAMILTON-JACOBI EQUATIONS

Michael G. Crandall and Pierre-Louis Lions

Introduction:

This paper introduces a new notion of solution for first order equations of Hamilton-Jacobi type (which we call HJ equations below). Attention will be focused on the following two classes of problems:

$$H(x,u,Du) = 0 \text{ in } \Omega, \quad u = z \text{ on } \partial\Omega, \tag{0.1}$$

which will be called the Dirichlet problem for HJ equations; and

$$\begin{aligned} &u_t + H(x,t,u,Du) = 0 && \text{in } \Omega \times]0,T], \\ &u = z \text{ on } \partial\Omega \times]0,T], \quad u(x,0) = u_0(x) && \text{in } \Omega, \end{aligned} \tag{0.2}$$

which will be called the Cauchy problem for HJ equations. Here and below Ω is any open domain in $\mathbb{R}^N$, z and u_0 are given functions (boundary conditions) and $H(x,u,p)$ (respectively, $H(x,t,u,p)$) is a given function on $\Omega \times \mathbb{R} \times \mathbb{R}^N$ (respectively, $\Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N$) which is called the Hamiltonian. The notation Du indicates the gradient of u with respect to the x variables: $Du = (u_{x_1}, \cdots, u_{x_N})$. We often take $\Omega = \mathbb{R}^N$ in which case the boundary condition z is replaced by requirements on the behaviour of u at ∞.

Problems (0.1), (0.2) are *global nonlinear first-order problems* and it is well-known that they do not have classical solutions - that is solutions $u \in C^1(\Omega)$ or $u \in C^1(\Omega \times]0,T])$ - in general, even if the Hamiltonian and boundary conditions are smooth. Thus these problems have been approached by looking for generalized solutions - usually solutions $u \in W^{1,\infty}_{loc}(\Omega)$ or $u \in W^{1,\infty}_{loc}(\Omega \times]0,T])$ - which satisfy the equations almost everywhere. In this context existence results have been obtained by several authors - e.g., A. Douglis [5], S. N. Kruzkov [18,19,20], W. H. Fleming [13,14,15], A. Friedman [16], S. H. Benton [4] with the most general results being given by P. L. Lions [22].

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The question of uniqueness of the solutions seems to be more difficult. The problems (0.1) and (0.2) may have many distinct generalized solutions. For example, if $\Omega = \mathbb{R}$, $\lambda > 0$, and $H(x,u,Du) = |u_x| + \lambda u - 1$, one checks easily that $u \equiv 1/\lambda$ is a classical solution of (0.1) while

$$u(x) = \begin{cases} \frac{1}{\lambda} - Ae^{\lambda x} & \text{for } x < x_0, \\ \frac{1}{\lambda} - Ae^{\lambda(2x_0 - x)} & \text{for } x > x_0, \end{cases}$$

is a bounded, Lipschitz continuous and piecewise analytic function which satisfies the equation except at $x = x_0$ for all choices of the parameters $A > 0$ and $x_0 \in \mathbb{R}$. Similarily, setting $\Omega = \mathbb{R}$, $u_0 \equiv 0$, $H(x,t,u,Du) = (u_x)^2$ in (0.2), we have the classical solution $u \equiv 0$ and the piecewise linear function

$$u = \begin{cases} 0 & \text{for } |x| > t > 0 \\ t - |x| & \text{for } t > |x|, \end{cases}$$

which satisfies the equation classically except on the lines $t = \pm x$, $x = 0$. In addition, if u,v are generalized solutions of (0.1) or (0.2) then so are $\min(u,v)$ and $\max(u,v)$. In fact, if the problems are nonlinear, one expects infinitely many $W^{1,\infty}_{loc}$ solutions (e.g., Conway and Hopf [6]).

The uniqueness problem is resolved in this paper by introducing a new notion of solution. We call these solutions <u>viscosity solutions</u>.(†) This notion of solution is given in Part I where we also develop basic results needed in the sequel. Later we establish, for each of the Cauchy and Dirichlet problems, uniqueness results for viscosity solutions. The question of existence in the class of viscosity solutions is also treated. This, however, usually reduces to checking that the standard existence mechanism provide viscosity solutions and passages to limits.

(†) This name refers to the "vanishing viscosity" method used in the existence results, and was chosen for want of a better idea.

The nature of the results is illustrated quite well by the following special case. Take (0.1) with $\Omega = \mathbb{R}^N$ and $H(x,u,p)$ replaced by $H(p) + u - n(x)$ where $H \in C(\mathbb{R}^N)$, $n \in BUC(\mathbb{R}^N)^{(\dagger)}$, i.e. (0.1) reads $H(Du) + u = n(x)$. In this case we take a viscosity solution of (0.1) to be a function $u \in C_b(\mathbb{R}^N)^{(\dagger)}$ which satisfies

$$(0.3)\quad \begin{cases} \forall \varphi \in C_0^\infty(\mathbb{R}^N),\ \varphi \geq 0,\ \forall k \in \mathbb{R} \text{ if } \max \varphi(u-k) > 0 \text{ (respectively,} \\ \min \varphi(u-k) < 0), \text{ then there exists } x_0 \in \{x: \varphi(u-k) = \max\varphi(u-k)\} \\ \text{(respectively, } \{x: \varphi(u-k) = \min\varphi(u-k)\}) \text{ such that} \\ H\left(-\left(\frac{(u-k)}{\varphi} D\varphi\right)(x_0)\right) + u(x_0) \leq n(x_0) \text{ (respectively, } \geq n(x_0)). \end{cases}$$

Under these assumptions, the results to follow imply:

(i) If u is a classical solution of (0.1), then u satisfies (0.3) (Section I).

(ii) If u is a viscosity solution of (0.1), and u is differentiable at some x_0, then $H(Du(x_0)) + u(x_0) = n(x_0)$; in particular, if u is locally Lipschitz then (0.1) holds a.e. (Section I).

(iii) If u,v are two viscosity solutions of (0.1), then $u \equiv v$ (Section II).

(iv) Let $\{H_m(p) + u - n_m\}$ be a sequence of Hamiltonians of the above form and u_m be a viscosity solution of the corresponding problem. If $H_m \to H$, $u_m \to u$, and $n_m \to n$ locally uniformly, then u satisfies (0.3). (Section I).

(v) The problem (0.1) has a viscosity solution u and $|u(x+y) - u(x)| \leq \sup\{|n(z+y) - n(z)| : z \in \mathbb{R}^N\}$. In particular, $u \in BUC(\mathbb{R}^N)$ and if $n \in C^{0,\alpha}(\mathbb{R}^N)$ then $u \in C^{0,\alpha}(\mathbb{R}^N)$, $0 < \alpha \leq 1$. (Section IV).

It is of interest here that the viscosity solution of (0.1) with $H(p) + u - n(x)$ as above exists and is unique in such generality. Indeed, the solution may be nowhere differentiable as is seen by taking $H \equiv 0$ and n to be nowhere differentiable. Thus we have a notion of solution of HJ equations which admits nowhere differentiable functions and permits a good existence and uniqueness theory. It is akin to the standard distribution theory, but "integration by parts" is replaced by "differentiation by parts"

(†) $BUC(\Omega)$ (respectively, $C_b(\Omega)$) denotes the space of bounded and uniformly continuous (respectively, bounded and continuous) functions on Ω.

and is done "inside" the nonlinearity. It is extremely convenient (as is the distribution theory) for passages to limits. The only somewhat related ideas we are aware of may be found in L. C. Evans [11], but there is also a definite parallel with the so-called "entropy condition" for scalar hyperbolic equations of the form $u_t + \sum f_i(u)_{x_i} = 0$. See E. Hopf [7], Vol'pert [26] and, especially, S. N. Kružkov [20].

Finally we recall that in the case of a convex Hamiltonian other uniqueness criteria are known (A. Douglis [10], S. N. Kružkov [18], P. L. Lions [22]). Some of the current results were announced in [8].

A few words about the presentation are in order. There are many interesting theorems in this subject. We have chosen what seem to us to be the most basic to discuss in some detail and then we make some remarks on variants. To keep the ideas clear we give a "layered" presentation - some proofs are given in simple cases and then more technical and general results are presented which subsume the simple ones. However, there is little redundancy, for we use the arguments given in the simple cases without repetition. Toward the end of the paper we give proofs in simple cases and refer the reader to previous arguments which show how to generalize. A first reading of this paper for the basic ideas could consist of Section I.1 and I.2 through Corollary I.6, Section II.1, Section IV and Sections V.1, V.2. We mention that some of these ideas obviously generalize to nonlinear second-order equations. The extent to which this is true is not yet clear and is being pursued by the authors. Another area for which the current results have implications is numerical approximation of HJ equations. Here we have obtained error estimates which show the convergence of a class of difference approximations to the viscosity solution [9].

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I. Viscosity Solutions.

I.1 Notation and Definitions.

Let $\mathcal{O}$ be an open set in $\mathbb{R}^M$ and $F(y,s,p)$ be a continuous function from $\mathcal{O} \times \mathbb{R} \times \mathbb{R}^M$ into $\mathbb{R}$. We consider the following equation

$$F(y,u,Du) = 0 \quad \text{in } \mathcal{O} . \tag{1.1}$$

where $Du = (u_{y_1}, \cdots, u_{y_M})$. We have in mind that (1.1) includes both (0.1) and (0.2) of the introduction. In the first case $\Omega = \mathcal{O}$ and $F = H$ while in the second $\mathcal{O} = \Omega \times]0,T[$, $y = (x,t)$ and $F(x,t,u,p) = p_{N+1} + H(x,t,p_1, \cdots, p_N)$.

If X is a set of functions on $\mathcal{O}$, then X^+ denotes the nonnegative functions in X and X_c denotes those functions in X which vanish off of a compact subset of $\mathcal{O}$. $\mathcal{D}(\Omega)$ denotes the C^∞ functions on $\mathcal{O}$ vanishing off a compact subset of $\mathcal{O}$, i.e. $\mathcal{D}(\Omega) = C_c^\infty(\mathcal{O})$. Convergence in $C(\mathcal{O})$ means uniform convergence on compact subsets of $\mathcal{O}$, etc.

To partially motivate the definitions to follow, consider a classical (i.e., C^1) solution u of (1.1). Let $\varphi \in C^1(\Omega)$ and $\varphi(y)u(y) = \max\varphi u > 0$. Then $D(\varphi u)(y) = \varphi(y)Du(y) + u(y)D\varphi(y) = 0$ or

$$Du(y) = -\frac{u(y)}{\varphi(y)} D\varphi(y) .$$

It follows that

$$F(y,u(y), -\frac{u(y)}{\varphi(y)} D\varphi(y)) = 0 .$$

We could do a similar computation at a positive maximum point y of $\varphi(u - \psi)$ where $\psi \in C^1(\Omega)$ as well to conclude

$$F(y,u(y), -\frac{(u(y) - \psi(y))}{\varphi(y)} D\varphi(y) + D\psi(y)) = 0 .$$

In the definitions which follow we specialize to $\psi \equiv k \in \mathbb{R}$.

We need some more notation. For $\psi \in C(\mathcal{O})$, set

$$E_+(\psi) = \{y \in \mathcal{O} : \psi(y) = \max \psi > 0\} \quad \text{(the positive extreme set of } \psi\text{)},$$

and

$$E_-(\psi) = \{y \in \mathcal{O} : \psi(y) = \min \psi < 0\} \quad \text{(the negative extreme set of } \psi\text{)},$$

with the understanding that $E_+(\psi) = \emptyset$ if ψ does not assume a positive maximum value in $\mathcal{O}$, etc. When necessary, the dependence on $\mathcal{O}$ will be recalled by writing $E_+(\psi;\mathcal{O})$, $E_-(\psi;\mathcal{O})$.

We now define viscosity solutions of (1.1) as well as the corresponding notions of sub- and super-solutions.

<u>Definition I.1.</u> A viscosity subsolution (respectively, supersolution) of (1.1) is a function $u \in C(\mathcal{O})$ such that for every $\varphi \in \mathcal{D}(\mathcal{O})^+$ and $k \in \mathbb{R}$

$$(1.2) \qquad \begin{cases} E_+(\varphi(u-k)) \neq \phi \implies \exists\, y \in E_+(\varphi(u-k)) \text{ such that} \\ F(y,u(y), -\dfrac{(u(y)-k)}{\varphi(y)} D\varphi(y)) \leq 0, \end{cases}$$

(respectively,

$$(1.3) \qquad \begin{cases} E_-(\varphi(u-k)) \neq \phi \implies \exists\, y \in E_-(\varphi(u-k)) \text{ such that} \\ F(y,u(y), -\dfrac{(u(y)-k)}{\varphi(y)} D\varphi(y)) \geq 0). \end{cases}$$

A viscosity solution is a $u \in C(\mathcal{O})$ for which both (1.2) and (1.3) hold, i.e. u is both a viscosity subsolution and a viscosity supersolution.

It will be convenient at times to speak of viscosity solutions of $F \leq 0$ rather than viscosity subsolutions of $F = 0$, etc. The reader should notice at this stage that the equations $F = 0$ and $-F = 0$ are <u>not</u> equivalent in the viscosity sense. For example, $u(x) = |x|$ is a viscosity solution of $(u_x)^2 - 1 = 0$ on $\mathbb{R}$, but it is not a viscosity solution of $-(u_x)^2 + 1 = 0$ on $\mathbb{R}$. (The reader can verify this as an exercise or turn to Section I.4.) However, we do have:

<u>Remark 1.4.</u> u is a viscosity solution $F(y,u,Du) \leq 0$ if and only if $v = -u$ is a viscosity solution of $-F(y, -v, -Dv) \geq 0$.

According to our "motivation", admittedly meager at this point, classical solutions are clearly viscosity solutions. Complete consistency of the classical and viscosity notions of solution requires that a viscosity solution u which happens to be C^1 also be a classical solution. This is indeed the case, as is a consequence of subtler facts presented in the next paragraph.

I.2. Basic Properties of Viscosity Solutions.

In this paragraph we develop a variety of basic results concerning viscosity solutions. A matter of concern will be showing that the weak assumptions in Definition I.1 - e.g., the small classes of functions $\varphi \in \mathcal{D}(\Omega)^+$, $\psi \equiv k \in \mathbb{R}$ occuring in the Definition as well as the "$\exists$" in place of "$\forall$" in (1.2), (1.3) - can be strengthened without altering the notion defined. Before stating results to this effect, we will prove one which illustrates the convenience of the weakness of the definition.

In order to set the stage for this result, we first give an example showing it to be totally false for Lipschitz continuous solutions. Consider the problem

$$\begin{cases} (u_x)^2 - 1 = 0 & \text{on} \quad]-1,1[\\ u(-1) = u(1) = 0 \ . \end{cases}$$

This problem has a largest Lipschitz solution $u_{max}(x) = 1 - |x|$ and a smallest Lipschitz solution $u_{min} = -u_{max}$. It has many others. E. g., $u_n(-1) = 0$, and $u_n' = (-1)^j$ on $]-1 + j/2n, -1 + (j+1)/2n[$ for $j = 0,\cdots,4n-1$, defines a solution for which $0 \leq u_n \leq 1/2n$ for each n. Clearly $u_n \to 0$ uniformly as $n \to \infty$, but $u \equiv 0$ is not a solution of $(u_x)^2 = 1$ anywhere. More generally, given any $g \in C([-1,1])$ with Lipschitz constant 1 and $g(-1) = g(1) = 0$, it can be uniformly approximated by Lipschitz continuous solutions of the above problem.

In contrast, for viscosity solutions we have:

<u>Theorem I.2.</u> (Stability of viscosity solutions). Let $\{F_\ell\}$ be a sequence of continuous functions on $\mathcal{O} \times \mathbb{R} \times \mathbb{R}^M$ converging in $C(\mathcal{O} \times \mathbb{R} \times \mathbb{R}^M)$ to $F \in C(\mathcal{O} \times \mathbb{R} \times \mathbb{R}^M)$ and let $u_\ell \in C(\mathcal{O})$ be a viscosity solution of $F_\ell(y,u_\ell,Du_\ell) \leq 0$ (respectively, $F_\ell \geq 0$). Let $u_\ell \to u$ in $C(\mathcal{O})$. Then u is a viscosity solution of $F \leq 0$ (respectively, $F \geq 0$).

<u>Proof of Theorem I.2.</u> Assume u_ℓ is a viscosity solution of $F_\ell \leq 0$. Let $\varphi \in \mathcal{D}(\mathcal{O})^+$ and $y \in E_+(\varphi(u-k))$. Then for large ℓ $\varphi(y)(u_\ell(y)-k) > 0$ so $E_+(\varphi(u_\ell-k)) \neq 0$ and, by assumption, there exists $y_\ell \in E_+(\varphi(u_\ell-k))$ for which

$$(1.5) \qquad F_\ell(y_\ell, u_\ell(y_\ell), -\frac{(u_\ell(y_\ell)-k)}{\varphi(y_\ell)} D\varphi(y_\ell)) \leq 0 .$$

Now $y_\ell \in \operatorname{supp} \varphi^{(\dagger)}$ and thus there is a subsequence $y_{\ell'}$ convergent to some $\bar{y} \in \mathcal{O}$. Moreover $\varphi(u-k) \leq \underline{\lim} \max(\varphi(u_\ell-k)) = \underline{\lim}\, \varphi(y_\ell)(u_\ell(y_\ell)-k) \leq \varphi(\bar{y})(u(\bar{y})-k)$ so $\bar{y} \in E_+(\varphi(u-k))$. Letting $\ell \to \infty$ through the subsequence ℓ' in (1.5) and using the assumed convergence $F_\ell \to F$ we have

$$F(\bar{y}, u(\bar{y}), -\frac{(u(\bar{y})-k)}{\varphi(\bar{y})} D\varphi(\bar{y})) \leq 0 .$$

Thus u is a viscosity subsolution. The proof for the case $F_\ell \geq 0$ is the same or one may use Remark 1.4. The proof is complete.

The next result summarizes the implications of the sequence of arguments which follow it and outlines the extent to which the definition of viscosity solution could be strengthened without changing the class of such solutions. If $\varphi \in C(\mathcal{O})$ we set

$$d(\varphi) = \{y \in \mathcal{O} : \varphi \text{ is differentiable at } y\}.$$

<u>Theorem I.3.</u> Let u be a viscosity subsolution of $F = 0$, $\varphi \in C(\mathcal{O})^+$ and $\psi \in C(\mathcal{O})$. Then

$$(1.6) \qquad F(y, u, -\frac{(u-\psi)}{\varphi} D\varphi + D\psi) \leq 0 \quad \text{on} \quad E_+(\varphi(u-\psi)) \cap d(\varphi) \cap d(\psi).$$

If u is a viscosity supersolution, then

$$(1.7) \qquad F(y, u, -\frac{(u-\psi)}{\varphi} D\varphi + D\psi) \geq 0 \quad \text{on} \quad E_-(\varphi(u-\psi)) \cap d(\varphi) \cap d(\psi),$$

while if u is a viscosity solution both (1.6) and (1.7) hold.

We prepare two lemmas. A key ingredient is the following formulation of a result of L. C. Evans [11].

(†) $\operatorname{Supp}\varphi$ denotes the support of φ.

Lemma I.4. Let $\varphi \in C(\mathcal{O})$ be differentiable at $y_0 \in \mathcal{O}$. Then there exist functions ψ_+ and ψ_- such that $\psi_\pm \in C_c^1(\mathcal{O})$, $\psi_\pm(y_0) = \varphi(y_0)$, $D\psi_\pm(y_0) = D\varphi(y_0)$ and $\psi_+ > \varphi$, $\psi_- < \varphi$ on $B(y_0,r) \setminus \{y_0\}$ (†) for some $r > 0$.

Proof of Lemma I.4. Replacing φ by $\hat{\varphi}(y) = \varphi(y_0 + y) - \varphi(y_0) - D\varphi(y_0) \cdot y$ (††) we can assume $y_0 = 0$, $\varphi(0) = 0$, and $D\varphi(0) = 0$. It suffices to exhibit ψ_+. By assumption, $\varphi(y) = |y|\rho(y)$ where $\rho \in C(\mathcal{O})$ and $\rho(y) \to 0$ as $|y| \to 0$. Set $\bar{\rho}(r) = \sup\{\rho(y) : y \in \mathcal{O} \cap B(0,r)\}$ and

$$\psi_+(y) = \int_{|y|}^{2|y|} \bar{\rho}(s)ds + |y|^2 .$$

Let $\overline{B(0,h)} \subset \mathcal{O}$. Then $\psi_+ \in C^1(B(0,h))$, $\psi_+(0) = 0$, $\psi_+(y) > |y|\bar{\rho}|y|) + |y|^2 > |y|\rho(y) = (y)$ for $y \in B(0,h)\setminus\{0\}$ by the monotonicity of $\bar{\rho}$, and $D\psi_+(0) = 0$. This ψ_+ may be modified outside $B(0,h/2)$ if necessary to achieve $\psi_+ \in C_c^1(\mathcal{O})$.

We next prove:

Lemma I.5. The assertions of Theorem I.3 are valid if also $\psi \equiv k \in \mathbb{R}$ is a constant.

Proof of Lemma I.5. It suffices to show (1.6) holds for viscosity supersolutions (recall Remark 1.4). Let $\varphi \in C(\mathcal{O})^+$ be differentiable at $y_0 \in \mathcal{O}$ and $y_0 \in E_+(\varphi(u-k))$. It follows at once from Lemma I.4 that there is a $\psi_- \in C_c^1(\mathcal{O})^+$ such that $\psi_-(y_0) = \varphi(y_0)$, $D\psi_-(y_0) = D\varphi(y_0)$ and $\psi_- < \varphi$ on $\operatorname{supp}\psi_- \setminus \{y_0\}$. Then

$$\{y_0\} = E_+(\psi_-(u-k)). \tag{1.8}$$

Next choose a sequence $\{\varphi_\ell\}_{\ell=1}^\infty \subset \mathcal{D}(\mathcal{O})^+$ with supports contained in a fixed compact subset of $\mathcal{O}$ so that $\varphi_\ell \to \psi_-$ and $D\varphi_\ell \to D\psi_-$ uniformly. For large ℓ, $\varphi_\ell(y_0)(u(y_0) - k) > 0$ so $E_+(\varphi_\ell(u-k)) \neq 0$ and, by assumption, there exist $y_\ell \in E_+(\varphi_\ell(u-k))$ such that

(†) $B(y_0,r)$ denotes the open ball of radius r and center y_0.

(††) $a \cdot b$ denotes the Euclidean inner-product of $a,b \in \mathbb{R}^N$.

$$(1.9) \qquad F(y_\ell, u(y_\ell), -\frac{(u(y_\ell)-k)}{\varphi(y_\ell)} D\varphi_\ell(y_\ell)) \leq 0 .$$

Passing to a subsequence if necessary we may assume y_ℓ converges to a limit y. Clearly $y \in E_+(\psi_-(u-k))$ and then $y = y_0$ by (1.8). Sending ℓ to ∞ in (1.9) and using $\varphi_\ell \to \psi_-$ in C^1, $\psi_-(y_0) = \varphi(y_0)$, $D\psi_-(y_0) = D\varphi(y_0)$ we conclude

$$F(y_0, u(y_0), -\frac{(u(y_0)-k)}{\varphi(y_0)} D\varphi(y_0)) \leq 0 ,$$

whence the result.

<u>Proof of Theorem I.3.</u> It suffices to consider the subsolution case. Let $\varphi \in C(\mathcal{O})^+$, $\psi \in C(\mathcal{O})$, $y_0 \in E_+(\varphi(u-\psi)) \cap d(\varphi) \cap d(\psi)$. Set

$$\tilde{\varphi}(y) = \varphi(y) \frac{u(y) - \psi(y)}{u(y) - \psi(y_0)} \chi(y)$$

where $\chi \in \mathcal{D}(\mathcal{O})^+$ satisfies $0 \leq \chi \leq 1$, $\chi(y_0) = 1$, and χ vanishes off a neighborhood of y_0 on which $u(y) > \psi(y_0)$. Then

$$\tilde{\varphi}(y)(u(y) - \psi(y_0)) = \chi(y)\varphi(y)(u(y) - \psi(y))$$

which is clearly at most $\varphi(y_0)(u(y_0) - \psi(y_0))$, i.e. $y_0 \in E_+(\tilde{\varphi}(u - \psi(y_0)))$. Since and ψ are differentiable at y_0 and

$$\frac{u(y) - \psi(y)}{u(y) - \psi(y_0)} = 1 + \frac{\psi(y_0) - \psi(y)}{u(y_0) - \psi(y_0) + u(y) - u(y_0)}$$

$$= 1 + \frac{\psi(y_0) - \psi(y)}{u(y_0) - \psi(y_0)} + o(|y-y_0|)$$

we have

$$D\tilde{\varphi}(y_0) = D\varphi(y_0) - \frac{1}{u(y_0) - \psi(y_0)} D\psi(y_0) .$$

The result now follows from Lemma I.5 applied with $k = \psi(y_0)$ and $\tilde{\varphi}$ in place of φ.

Using the above results it is now simple to prove:

Corollary I.6. (Consistency) Let u be a viscosity subsolution (respectively; supersolution, solution) of $F(y,u,Du) = 0$. Then $F(y,u,Du) \leq 0$ (respectively; $F(y,u,Du) \geq 0$, $F(y,u,Du) = 0$) on $d(u)$.

Proof of Corollary I.6. It suffices to treat the supersolution case. Let $y_0 \in d(u)$. Choose $\psi_+ \in C_c^1(\mathcal{O})$ such that $\psi_+(y_0) = u(y_0)$, $D\psi_+(y_0) = Du(y_0)$ and $\psi_+ > u$ in a deleted ball $B(y_0,h) \setminus \{y_0\}$. Choose $\varphi \in \mathcal{D}(\mathcal{O})^+$ with $\operatorname{supp}\varphi \subset B(y_0,h)$, $0 \leq \varphi \leq 1$, $\varphi(y_0) = 1$ (so $D\varphi(y_0) = 0$). Then $\{y_0\} = E_-(\varphi(u - \psi_+ + 1))$. By Theorem I.3 and the assumption that u is a viscosity supersolution, we have

$$F\left(y_0, u(y_0), -\frac{(u(y_0)-\psi_+(y_0)+1)}{\varphi(y_0)} D\varphi(y_0) + D\psi_+(y_0)\right) = F(y_0,u(y_0),Du(y_0)) \geq 0 ,$$

and the proof is complete.

The next two results are concerned with changes of variables.

Corollary I.7. Let u be a viscosity subsolution (respectively, supersolution, solution) of (1.1). Then:

(i) If $g \in C^1(\mathcal{O})$, $g > 0$ in $\mathcal{O}$, $\psi \in C^1(\mathcal{O})$ and $v = g(u-\psi)$, then v is a viscosity solution (respectively; supersolution, solution) of $G(y,v,Dv) = 0$ where

$$G(y,r,p) = F\left(y, \frac{r}{g(y)} + \psi(y), \frac{-rDg(y)}{g(y)^2} + \frac{p}{g(y)} + D\psi(y)\right).$$

(ii) If $\Phi : \mathcal{O} \to \hat{\mathcal{O}}$ is a C^1 diffeomorphism of the domain $\mathcal{O}$ onto the domain $\hat{\mathcal{O}}$, Then $v(\Phi(y)) = u(y)$ defines a viscosity subsolution (respectively; supersolution, solution) of $G(\hat{y},v,Dv) = 0$ where

$$G(\hat{y},r,p) = H(\Phi^{-1}(\hat{y}),r,pD\Phi(\Phi^{-1}(\hat{y}))$$

and $pD\Phi(y)$ denotes the action of $D\Phi(y)$ on the cotangent vector p.

We omit the proof of Corollary I.7 as it is an easy exercise given Theorem I.3. To conclude this section we obtain a partial result concerning nonlinear changes of the unknown.

Corollary I.8. Let u be a viscosity subsolution (respectively; supersolution, solution) of (1.1) and let $\Phi \in C^1(\mathbb{R})$, $\Phi' > 0$ everywhere and $\Phi(\mathbb{R}) = \mathbb{R}$. Then $v = \Phi(u)$ is a viscosity subsolution (respectively; supersolution, solution) of

$$F(y, \Phi^{-1}(v), (\Phi^{-1})'(v)Dv) = 0 \ . \tag{1.10}$$

Proof of Corollary I.8. We treat the subsolution case. Let u be a viscosity subsolution of $F = 0$. We claim that, if $x_0 \in E_+(\varphi(v-k))$ (with $\varphi \in \mathcal{D}(\Omega)^+$, $k \in \mathbb{R}$) then there exists $\tilde{\varphi} \in C_c^1(\Omega)^+$, $\tilde{k} \in \mathbb{R}$ such that

$$x_0 \in E_+(\tilde{\varphi}(u-\tilde{k})), \ -\frac{(u(x_0)-\tilde{k})}{\tilde{\varphi}(x_0)} D\tilde{\varphi}(x_0) = -\Psi'(v(x_0)) \frac{v(x_0)-k}{\varphi(x_0)} D\varphi(x_0),$$

where $\Psi(t) = \Phi^{-1}(t)$. This obviously implies the Corollary.

Now, to prove our claim, we argue as follows: we have for $|x-x_0|$ small

$$\begin{aligned} v(x) &\leq \frac{\varphi(x_0)}{\varphi(x)} (v(x_0)-k) + k \\ &\leq v(x_0) - \frac{(v(x_0)-k)}{\varphi(x_0)} D\varphi(x_0)\cdot(x-x_0) + |x-x_0|\varepsilon(|x-x_0|) \end{aligned} \tag{1.11}$$

where $\varepsilon \in C(\mathbb{R}_+, \mathbb{R}^M)$ and $\varepsilon(t) \to 0$ as $t \to 0+$. Thus, for $|x-x_0|$ small, we obtain since Ψ is nondecreasing

$$\begin{aligned} u(x) &\leq \Psi(v(x_0) - \left(\frac{v(x_0)-k}{\varphi(x_0)}\right) D\varphi(x_0)\cdot(x-x_0) + |x-x_0|\varepsilon(|x-x_0|)) \\ &\leq \tilde{u}(x) = u(x_0) - \Psi'(v(x_0))\left(\frac{v(x_0)-k}{\varphi(x_0)}\right) D\varphi(x_0)\cdot|x-x_0| + |x-x_0|\tilde{\varepsilon}(|x-x_0|) \end{aligned}$$

for $|x-x_0|$ small enough and $\tilde{\varepsilon} \in C(\mathbb{R}_+, \mathbb{R}^M)$, $\tilde{\varepsilon}(t) \to 0$ as $t \to 0+$. But the right hand side member $\tilde{u}$ of the above inequality is a continuous function differentiable at x_0 and therefore by Lemma I.4 we may find $\tilde{k}$ and $\tilde{\varphi} \in C_c^1(\Omega)^+$ such that: $u(x_0) - \tilde{k} > 0$; $x_0 \in E_+(\tilde{\varphi}(\tilde{u}-\tilde{k}))$

$$-D\tilde{\varphi}(x_0) \frac{u(x_0)-\tilde{k}}{\tilde{\varphi}(x_0)} = -\Psi'(v(x_0)) \frac{v(x_0)-k}{\varphi(x_0)} D\varphi(x_0); \quad \operatorname{supp}\tilde{\varphi} \subset B(x_0, h)$$

where h is small enough in order to have: $u(x) \leq \tilde{u}(x)$ on $B(x_0,h)$. We may now conclude since we have for all x:

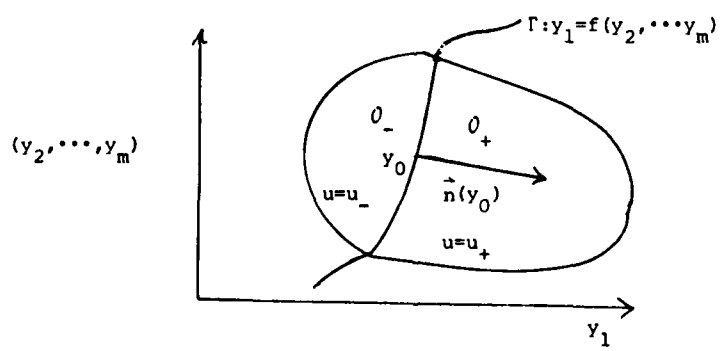
$$\tilde{\varphi}(x)(u(x)-\tilde{k}) \leq \tilde{\varphi}(x)(\tilde{u}(x)-\tilde{k})\tilde{\varphi}(x_0)(\tilde{u}(x_0)-\tilde{k}) = \tilde{\varphi}(x_0)(u(x_0)\tilde{k}) \tag{1.12}$$

and thus $x_0 \in E_+(\tilde{\varphi}(u-\tilde{k}))$.

<u>Remark 1.13.</u> We pause here to consider the case in which $\mathcal{O}$ is not an open subset of $\mathbb{R}^N$. Indeed, in later sections we will want to use some of the above results when $\mathcal{O}$ has the form $\mathcal{O} = \Omega \times (]0,T])$. We claim that all we have done is correct in general if one interprets the definitions appropriately. This means: $\mathcal{D}(\mathcal{O})$, $C^1(\mathcal{O})$, etc., should denote restrictions of functions in $\mathcal{D}(\mathbb{R}^N)$, $C^1(\mathbb{R}^N)$, etc. to $\mathcal{O}$ (with, in the case of $\mathcal{D}(\mathcal{O})$, $\{x \in \mathcal{O}: u(x) \neq 0\}$ lies in a compact subset of $\mathcal{O}$, etc). The other point is the notion of "differentiable". We will say $\varphi \in C(\mathcal{O})$ is differentiable at $y_0 \in \mathcal{O}$ and $D\varphi(y_0) = z$ if there is an extension of φ to $\tilde{\varphi} \in C(\mathbb{R}^N)$ such that $D\tilde{\varphi}(y_0) = z$ and moreover, for any extension of φ to $\tilde{\varphi} \in C(\mathbb{R}^N)$ differentiable at y_0, $D\tilde{\varphi}(y_0) = z$. (In the case where φ has some boundary which is sufficiently smooth, e.g. $\varphi = \Omega \times]0,T]$, all notions coincide.) The reader can think through these claims.

I.3. <u>Piecewise Smooth Viscosity Solutions.</u>

In this section we consider piecewise C^1 functions and determine conditions on the discontinuities of their derivatives equivalent to being viscosity solutions of $F = 0$. Consider the situation in Figure 1:



which is meant to indicate that $\mathcal{O} = \mathcal{O}_+ \cup \mathcal{O}_- \cup \Gamma$ is divided into two open parts $\mathcal{O}_+$ and $\mathcal{O}_-$ by the surface Γ. The unit normal to Γ at $y_0 \in \Gamma$ is $\hat{n}(y_0)$ and it points into $\mathcal{O}_+$. A function $u \in C(\mathcal{O})$ is given as u_+ in $\mathcal{O}_+ \cup \Gamma$ and u_- in $\mathcal{O}_- \cup \Gamma$. We assume Γ is of class C^1 and so may be represented by a relation of the typical form $y_1 = f(y_2,\cdots,y_m)$ near $y_0 \in \Gamma$, where $f \in C^1$. We assume $u \in C(\mathcal{O})$ and $u_\pm \in C^1(\mathcal{O}_\pm \cup \Gamma)$. When is u a viscosity solution of $F = 0$ in $\mathcal{O}$? We will use the following observations:

<u>Proposition I.9.</u> (i) If u is a viscosity solution of $F = 0$ in $\mathcal{O}$ and $\mathcal{O}'$ is an open subset of $\mathcal{O}$ then $u|_{\mathcal{O}'}$ (†) is a viscosity solution of $F = 0$ in $\mathcal{O}'$.
(ii) If $u \in C(\mathcal{O})$, $\mathcal{O}$ is the union of relatively open subsets $\mathcal{O}_1$ and $\mathcal{O}_2$, $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$ and $u|_{\mathcal{O}'}$ is a viscosity solution of $F = 0$ in $\mathcal{O}_i$, $i = 1,2$ then u is a viscosity solution of $F = 0$ in $\mathcal{O}$.

That is, the property of being a viscosity solution is purely local. Part (i) of the Proposition is completely trivial and we leave part (ii) as a very simple exercise.

To continue, assume $u \in C(\mathcal{O})$ is a viscosity solution. Then $u_\pm$ is a viscosity solution in $\mathcal{O}_\pm$. But $u_\pm$ lie in $C^1(\mathcal{O}_\pm)$, so $u_\pm$ are classical solutions by Corollary I.6. Let $\varphi \in \mathcal{D}(\mathcal{O})^+$, $y_0 \in E_\pm(\varphi(u-k))$. If $y_0 \in \mathcal{O}_+ \cup \mathcal{O}_-$ we then have

$$F(y_0,u(y_0), -(u(y_0)-k)\frac{D\varphi(y_0)}{\varphi(y_0)}) = 0$$

by the opening remarks of this section. It remains to consider $y_0 \in \Gamma$, and we assume Figure 1. Let

$$T_{y_0} = \{\tau \in R^M : \hat{n}(y_0) \cdot \tau = 0\}$$

be the tangent space to Γ to y_0 and p_T, $p_N = I - p_T$ be the orthogonal projections

(†) $u|_{\mathcal{O}'}$ means the restriction of u to $\mathcal{O}'$.

on T_{y_0}, $\mathrm{span}\{\vec{n}(y_0)\}$, i.e. $p_N y = (\vec{n}(y_0) \cdot y)\vec{n}(y_0)$. Since u_+, u_- agree on Γ , $p_T\, Du_+(y_0) = p_T\, Du_-(y_0)$. When $y_0 \in E_+(\varphi(u-k)) \cap \Gamma$ we clearly have:

$$T_{y_0} \ni \tau \to \Phi(\tau) = \varphi(y_0+\tau)(u(y_0+\tau)-k) \quad \text{satisfies} \quad D_T\Phi(0) = 0 ,$$

$$\lim_{\alpha\downarrow 0} \frac{\varphi(y_0+\alpha\vec{n})(u_+(y_0+\alpha\vec{n})-k) - \varphi(y_0)(u(y_0)-k)}{\alpha} \leq 0$$

$$\lim_{\alpha\uparrow 0} \frac{\varphi(y_0+\alpha\vec{n})(u_-(y_0+\alpha\vec{n})-k) - \varphi(y_0)(u(y_0)-k)}{\alpha} \geq 0 .$$

These relations amount to:

$$-\frac{(u(y_0)-k}{\varphi(y_0)}\, p_T D\varphi(y_0) = p_T Du_+(y_0) = p_T Du_-(y_0) ,$$

$$-\frac{(u(y_0)-k)}{\varphi(y_0)}\, D\varphi(y_0) \cdot \vec{n}(y_0) \geq Du_+(y_0) \cdot \vec{n}(y_0)$$

$$-\frac{(u(y_0)-k)}{\varphi(y_0)}\, D\varphi(y_0) \cdot \vec{n}(y_0) \leq Du_-(y_0) \cdot \vec{n}(y_0).$$

Hence

$$\begin{cases} -\dfrac{(u(y_0)-k)}{\varphi(y_0)}\, D\varphi(y_0) = p_T Du_-(y_0) + \xi\vec{n}(y_0) \\[2ex] \text{for some } \xi \in [Du_+(y_0) \cdot \vec{n}(y_0),\ Du_-(y_0) \cdot \vec{n}(y_0)] . \end{cases}$$

We conclude that the condition

$$(1.14) \qquad \begin{cases} \forall\, y_0 \in \Gamma,\ \forall\, \xi \in [Du_+(y_0) \cdot \vec{n}(y_0),\ Du_-(y_0) \cdot \vec{n}(y_0)] , \\[2ex] F(y_0, u(y_0),\ p_T Du_\pm(y_0) + \xi\vec{n}(y_0)) \leq 0 \end{cases}$$

implies u is a viscosity subsolution of $F = 0$. Similarly

$$(1.15) \qquad \begin{cases} \forall\, y_0 \in \Gamma,\ \forall\, \xi \in [Du_-(y_0) \cdot \vec{n}(y_0),\ Du_+(y_0) \cdot \vec{n}(y_0)] , \\[2ex] F(y_0, u(y_0),\ p_T Du_\pm(y_0) + \xi\vec{n}(y_0)) \geq 0 \end{cases}$$

implies u is a viscosity supersolution of $F = 0$. Note that if, e.g., $Du_-(y_0) \cdot \vec{n}(y_0) > Du_+(y_0) \cdot \vec{n}(y_0)$ then (1.15) is an empty condition, etc. In fact, (1.14), (1.15) are necessary as well as sufficient. We prove:

<u>Theorem I.10.</u> Let $\mathcal{O}$, $\mathcal{O}_+$, $\mathcal{O}_-$, Γ, u, $u_\pm$ be as above. Then u is a viscosity solution of $F = 0$ in $\mathcal{O}$ if and only if $u_\pm$ are classical solutions in $\mathcal{O}_\pm$ and (1.14), (1.15) hold.

<u>Proof.</u> The sufficiency has been shown. We consider the necessity. First let $\xi = DU_+(y_0) \cdot \vec{n}(y_0) = Du_-(y_0) \cdot \vec{n}(y_0)$. In this case u is differentiable at y_0 and $Du(y_0) = p_T\, Du_\pm(y_0) + \xi\vec{n}(y_0)$. By Theorem I.2 we have

$$F(y_0, u(y_0), Du(y_0)) = F(y_0, u(y_0), p_T Du_\pm(y_0) + \xi\vec{n}(y_0)) = 0$$

so (1.14), (1.15) hold. Next assume that $Du_-(y_0) \cdot \vec{n}(y_0) > \xi > Du_+(y_0) \cdot \vec{n}(y_0)$. We claim that then there is a $\psi \in C^1(\mathcal{O})$ such that $\psi(y_0) = u(y_0)$, $\psi > u$ in a deleted neighborhood of y_0 and $D\psi(y_0) = p_T\, Du_\pm(y_0) + \xi\vec{n}(y_0)$. If this is so, choose $\varphi \in \mathcal{D}(\mathcal{O})$, $0 \le \varphi \le 1$, $\varphi(y_0) = 1$ and $\varphi(y) < 1$ for $y \ne y_0$ so that $1 > \varphi(\psi - u) \ge 0$. Then $\{y_0\} = E_+(\varphi(u-\psi+1))$ and by Theorem I.3

$$F(y_0, u(y_0), D\psi(y_0)) = F(y_0, u(y_0), p_T Du_\pm(y_0) + \xi\vec{n}(y_0)) \le 0 ,$$

so we have (1.14). The case in which (1.14) is an empty requirement is similar. It remains to exhibit ψ. By Proposition I.9 and Corollary I.7 we may localize and change variables. Hence assume $y_0 = 0$ and Γ is $y_1 = 0$. We have

$$u(y_1, \cdots, y_m) = \begin{cases} u_+(y_1, \cdots, y_m) & \text{if } y_1 \ge 0 \\ u_-(y_1, \cdots, y_m) & \text{if } y_1 < 0 , \end{cases}$$

and

$$\frac{\partial u_-}{\partial y_i}(0, y_2, \cdots, y_m) = \frac{\partial u_+}{\partial y_i}(0, y_2, \cdots, y_m) \qquad i = 2, \cdots, m ,$$

$$\frac{\partial u_+}{\partial y_1}(0, 0, \cdots, 0) < \xi < \frac{\partial u_-}{\partial y_1}(0, 0, \cdots, 0) .$$

Let $\psi_0(y_2,\cdots,y_m) > u_\pm(0,y_2,\cdots,y_m)$ with strict inequality if $(y_2,\cdots,y_m) \neq (0,\cdots,0)$ in some neighborhood of $(0,\cdots,0)$, $\psi_0(0,\cdots,0) = u_\pm(0,0,\cdots,0)$, $\frac{\partial\psi_0}{\partial y_i}(0,\cdots,0) = \frac{\partial u_\pm}{\partial y_i}(0,\cdots 0)$ for $i = 2,\cdots,m$. ψ_0 exists by Lemma I.4. Then set $\psi(y_1,\cdots,y_m) = \psi_0(y_2,\cdots,y_m) + \xi y_1$. Clearly ψ has the desired properties and the proof is complete.

To illustrate this result, consider the example solution $u = 0$ for $|x| > t > 0$ $u = t - |x|$ if $|x| \leq t$ of $u_t + (u_x)^2 = 0$ in the introduction. Let Γ be $x = 0$, $\vec{n}(0,t) = (1,0)$. Then $F((x,t),u,(p_1,p_2)) = p_2 + (p_1)^2$, $u_+ = t - x$ and $u_- = t + x$ in the appropriate domains. We have

$$\begin{cases} p_T u_\pm(0,t) = (0,1) \\ Du_+(0,t) \cdot \vec{n}(0,t) = -1 < 1 = Du_-(0,t) \cdot \vec{n}(0,t) \end{cases}$$

but $F(p_T u_\pm(0,t) + \xi\vec{n}(0,t)) = 1 + \xi^2 > 0$ for $-1 < \xi < 1$ so (1.14) fails.

We remark that the conditions (1.14), (1.15) were anticipated by Oleinik [24] in a special case. Moreover, an alternative way to obtain these results is given in Section I.5.

I.4. Differential Inequalities in the Viscosity Sense.

In this section we treat some elementary inequalities in the viscosity sense. The first result concerns the one dimensional case.

Proposition I.11. Let $T > 0$ and $g,h \in C([0,T])$. Assume g is a viscosity solution of

(1.16) $$g' \leq h$$

in $]0,T[$. Then

(1.17) $$g(t) \leq g(s) + \int_s^t h(\tau)d\tau \quad \text{for } 0 \leq s \leq t \leq T .$$

Proof. It is enough to show (1.17) for $s = 0$ and for this it suffices to prove that for $\varepsilon > 0$

$$g(t) \leq g(0) + \int_0^t h(s)ds + \varepsilon + \varepsilon t , \quad 0 \leq t \leq T . \tag{1.18}$$

Assume (1.18) is false and let $\bar{t} \in]0,T[$ be the least t for which equality holds in (1.18). Set $\psi(t) = g(0) + \int_0^t h(s)ds + \varepsilon$ and note $\psi(0) > g(0)$, $\psi(\bar{t}) < g(\bar{t})$. Choose $\delta > 0$ such that $\psi(t) > g(t)$ on $[0,\delta]$ and $\eta \in C^1([0,T])^+$ such that $\eta' < 0$ on $[\delta,T]$ and $\eta(T) = 0$. Then there is a $t_0 \in E_+(\eta(g-\psi))$ and $t_0 \in]\delta,T[$. By Theorem I.3

$$-\frac{\eta'(t_0)}{\eta(t_0)}(g(t_0) - \psi(t_0)) + \psi'(t_0) \leq h(t_0).$$

Since $\eta'(t_0) < 0$ we have

$$\psi'(t_0) = h(t_0) < h(t_0)$$

which is a contradiction.

Remark 1.19. It follows from Proposition I.11 that (1.16) holds in the viscosity sense if and only if it holds in the sense of distributions.

Corollary I.12. Let $T > 0$, $\gamma \in \mathbb{R}$ and $g,h \in C([0,T])$. Let g be a viscosity solution of

$$g' + \gamma g \leq h \quad \text{on} \quad]0,T[. \tag{1.20}$$

Then

$$e^{\gamma t}g(t) \leq e^{\gamma s}g(s) + \int_s^t e^{\gamma \tau}h(\tau)d\tau \quad \text{for} \quad 0 \leq s \leq t \leq T . \tag{1.21}$$

Proof. By Remark 1.19, (1.2) holds in the sense of distributions and then it is known that (1.21) holds. (Of course, one could prove (1.21) directly by adapting the proof of the Proposition or by using Corollary I.7 to find $(e^{\gamma t}g)' \leq e^{\gamma t}h$ in the viscosity sense.)

In the next result we show that u is a viscosity subsolution of

$$\frac{\partial}{\partial y_1} u(y_1,y_2,\cdots,y_m) = g(y_1,\cdots,y_m) \tag{1.22}$$

exactly when the corresponding statement holds for the functions of one variable $r \mapsto u(r,y_2,\cdots,y_m)$ obtained by fixing $(y_2,\cdots,y_m)$.

Proposition I.14. Let $u, g \in C(\mathcal{O})$. For $z = (y_2, \cdots, y_m) \in R^{M-1}$, let $\mathcal{O}_z = \{r : (r,z) \in \mathcal{O}\}$. Let $u_z(r) = u(r,z)$, $g_z(r) = g(r,z)$ on $\mathcal{O}_z$. Then the following are equivalent:

$$(1.23) \quad \begin{cases} \text{For each } z \in R^{M-1}, \ u_z \text{ is a viscosity solution of} \\ \qquad u_z' \leq g_z \quad \text{in} \quad \mathcal{O}_z . \end{cases}$$

$$(1.24) \quad \begin{cases} u \text{ is a viscosity subsolution of} \\ \dfrac{\partial}{\partial y_1} u(y_1, \cdots, y_m) = g(y_1, \cdots, y_m) \quad \text{in} \quad \mathcal{O}. \end{cases}$$

Proof. We show (1.24) implies (1.23). Let $z_0 \in R^{m-1}$ be such that $\mathcal{O}_{z_0} \neq \phi$. Let $\eta \in \mathcal{D}(\mathcal{O}_{z_0})^+$, $k \in R$ and $r_0 \in E_+(\eta(u_{z_0} - k) : \mathcal{O}_{z_0})$. Using Lemma I.4 in the usual way we may assume $\{r_0\} = E_+(\eta(u_{z_0} - k) : \mathcal{O}_{z_0})$. Pick $\varphi \in \mathcal{D}(B(z_0, 1))^+$ such that $\varphi(z_0) = 1$. Set $\varphi_\varepsilon(z) = \varphi(z/\varepsilon)$. For $\varepsilon > 0$ and small, $\eta(y_1)\varphi_\varepsilon(y_2, \cdots, y_m) \in \mathcal{D}(\mathcal{O})^+$ and there exists $(r_\varepsilon, z_\varepsilon) \in E_+(\eta\varphi(u-k) : \mathcal{O})$. By assumption,

$$(1.25) \qquad -\frac{\eta'(r_\varepsilon)}{\eta(r_\varepsilon)} (u(r_\varepsilon, z_\varepsilon) - k) \leq g(r_\varepsilon, z_\varepsilon).$$

Clearly $z_\varepsilon \to z_0$ and $r_\varepsilon \to r_0$ as $\varepsilon \downarrow 0$. Thus the result follows by letting $\varepsilon \downarrow 0$ in (1.25).

It remains to show that (1.23) implies (1.24). However, this amounts to checking the definitions and is left to the reader.

The next result is concerned with more general directional derivatives.

Theorem I.15. Let $v : \mathcal{O} \to R^M$ be continuously differentiable. Denote by $Y(\tau, y_0)$ The solution of

$$(1.26) \quad \begin{cases} \dfrac{dY}{d\tau} = v(Y) , \\ Y(0, y_0) = y_0 , \end{cases}$$

which is defined on a maximal interval of existence I_{y_0}. (By assumption $Y(I_{y_0},y_0) \subset \mathcal{O}$). Let $u,g \in C(\mathcal{O})$ and u be a viscosity solution of

(1.27) $$(Du) \cdot \nu \leq g \quad \text{in} \quad \mathcal{O}.$$

Then for $y_0 \in \mathcal{O}, s,t \in I_{y_0}$ and $s \leq t$ one has

(1.28) $$u(Y(t,y_0)) - u(Y(s,y_0)) \leq \int_s^t g(Y(\tau,y_0))d\tau .$$

<u>Proof.</u> If $\nu(y_0) = 0$, then $Y(\tau,y_0) \equiv y_0$ and there is nothing to show. If $\nu(y_0) \neq 0$, we may rotate coordinates so that $\nu(y_0) = (\nu_1(y_0),0,\cdots,0)$. Without loss of generality we also assume $y_0 = 0$. Consider the change of variables Φ defined near $y_0 = 0$ by

$$\Phi(y_1,\cdots,y_m) = (\hat{y}_1,\cdots,\hat{y}_m) \iff (y_1,\cdots,y_m) = Y(\hat{y}_1,(0,\hat{y}_2,\cdots,\hat{y}_m)).$$

Then, with the notation of Corollary I.7 and $H(y,r,p) = p \cdot \nu(y) - g(y)$, we have

$$G(\hat{y},r,p) = pD\Phi(\Phi^{-1}(\hat{y})) \cdot \nu(\Phi^{-1}(\hat{y})) - g(\Phi^{-1}(\hat{y}))$$

$$= p_1 - g(\Phi^{-1}(\hat{y})).$$

(Of course, this is merely the statement that $\partial/\partial\hat{y}_1 = \nu \cdot (\partial/\partial y_1,\cdots,\partial/\partial y_m)$). Thus, by Corollary I.7, $u(\Phi^{-1}(\hat{y}))$ is a viscosity solution of

$$\frac{\partial}{\partial y_1} u \leq g(\Phi^{-1}(\hat{y})).$$

Propositions I.14 and I.11 then yield

$$u(\Phi^{-1}(t,0,\cdots,0)) - u(\Phi^{-1}(s,0,\cdots,0) \leq \int_s^t g(\Phi^{-1}(\tau,0,\cdots,0))d\tau$$

for $s \leq t$ and $|s|,|t|$ small. But this means

$$u(I(t,0)) - u(Y(s,0)) \leq \int_s^t g(Y(\tau,0))d\tau .$$

While this inequality is only established for $|s|,|t|$ small, it is then trivially extendable to $t,s \in I_0$, $s \leq t$.

<u>Corollary I.16.</u> Let $\mathcal{O}$ be convex, $u \in C(\mathcal{O})$ and $L \in \mathbf{R}$. If for every $\varphi \in \mathcal{D}(\mathcal{O})^+$ and $k \in \mathbf{R}$

$$\frac{(u-k)}{\varphi}\,|D\varphi| \leq L \quad \text{on} \quad E_+(\varphi(u-k)) \tag{1.29}$$

then $|u(\bar{y}) - u(\hat{y})| \leq L|\bar{y} - \hat{y}|$ for $y, \hat{y} \in \mathcal{O}$.

<u>Proof.</u> Fix $y, \hat{y} \in \mathcal{O}$ with $y \neq \hat{y}$. Put $\nu \equiv (|\bar{y} - \hat{y}|)^{-1}(\bar{y} - \hat{y})$. From (1.29) it follows that u is a viscosity solution of $Du \cdot \nu \leq L$ in $\mathcal{O}$. By Theorem I.15

$$u(y_0 + t\nu) - u(y_0 + s\nu) \leq \int_s^t M d\tau = M(t-s)$$

whenever $s \leq t$ and $y_0, y_0 + t\nu, y_0 + s\nu \in \mathcal{O}$. Set $y_0 = \hat{y}$, $t = |\bar{y} - \hat{y}|$, $s = 0$ to obtain $u(\bar{y}) - u(\hat{y}) \leq L|\bar{y} - \hat{y}|$. Since we may interchange $\bar{y}$ and $\hat{y}$, the proof is complete.

I.5. Characterization of Points in Some $E_+(\varphi(u-\psi))$.

According to Theorem I.3, if u is a viscosity solution of $F \leq 0$, then

$$F(y,u, -\frac{(u-\psi)}{\varphi} D\varphi+D\psi) \leq 0 \quad \text{on} \quad E_+(\varphi(u-\psi)) \cap d(\varphi) \cap d(\psi) .$$

One is naturally led to ask: What are the points y belonging to some $E_+(\varphi(u-\psi)) \cap d(\varphi) \cap d(\psi)$ and what are the possible values of $-(\frac{(u-\psi)}{\varphi}(D\varphi)) + D\psi$ at such points? We prove:

Theroem I.17. Let $u \in C(\mathcal{O})$ and $y_0 \in \mathcal{O}$, $a \in R^M$. Then the problem

$$(1.30) \qquad \begin{cases} y_0 \in E_+(\varphi(u-\psi)) \cap d(\varphi) \cap d(\psi) , \\ -\dfrac{(u(y_0) - \psi(y_0))}{\varphi(y_0)} D\varphi(y_0) + D\psi(y_0) = a \end{cases}$$

has a solution $\varphi \in C(\mathcal{O})^+$, $\psi \in C(\mathcal{O})$ if and only if there exists $\tilde{\psi} \in C^1(\mathcal{O})$ such that $\tilde{\psi}(y_0) = u(y_0)$, $\tilde{\psi} > u$ near y_0 and $D\tilde{\psi}(y_0) = a$. If E_+ is replaced by E_- in (1.30) and $\tilde{\psi} > u$ is replaced by $\tilde{\psi} < u$, the statement remains true.

Proof. We first observe the sufficiency. Let $\tilde{\psi} \in C^1(\mathcal{O})$ and $\tilde{\psi} < u$ near y_0. Choose $\varphi \in C^1(\mathcal{O})^+$ with a strict maximum value of 1 at y_0 and $\operatorname{supp}\varphi \subset \{\tilde{\psi} < u\}$. Then

$$y_0 \in E_+(\varphi(u - \tilde{\psi} + 1)) \quad \text{and} \quad -\frac{(u(y_0)-\tilde{\psi}(y_0)+1)}{\varphi(y_0)} D\varphi(y_0) + D\tilde{\psi}(y_0) = D\tilde{\psi}(y_0) \quad \text{since} \quad D\varphi(y_0) = 0.$$

The necessity is equally simple. Since $y_0 \in E_+(\varphi(u-\psi)) \cap d(\varphi) \cap d(\psi)$ implies

$$u(y) \leq \frac{1}{\varphi(y)} (\varphi(y_0)(u(y_0) - \psi(y_0)) + \psi(y)$$

near y_0 and the right-hand side is differentiable at y_0 with the derivative

$$-\frac{(u(y_0) - \psi(y_0))}{\varphi(y_0)} D\varphi(y_0) + D\psi(y_0) ,$$

we may majorize it near y_0 by a $\tilde{\psi} \in C^1(\mathcal{O})$ which agrees to first order at y_0 (Lemma I.4). This completes the proof.

Remark (1.31). By Lemma I.4 we may equally well characterize the pairs (y_0,a) for which (1.30) has a solution by the condition

$$\lim_{y\to y_0} \frac{\max\{(u(y) - (u(y_0) + a \cdot (y - y_0)),0\}}{|y - y_0|} = 0 .$$

Corollary I.18. Let $u \in C(\mathcal{O})$. Then

$$A_+ = \{y_0 \in \mathcal{O}: \exists\, \tilde{\psi} \in C^1(\mathcal{O}),\ \tilde{\psi}(y_0) = u(y_0) \text{ and } \tilde{\psi} > u \text{ near } y_0\}$$

is dense in $\mathcal{O}$. Similarly, the set A_- defined as above with $\tilde{\psi} > u$ replaced by $u > \tilde{\psi}$ is dense in $\mathcal{O}$.

Proof. If $y_0 \in \mathcal{O}$ and $\varepsilon > 0$, choose $\varphi \in C_c^1(\mathcal{O})^+$ so that $\varphi(y_0) > 0$ and $\operatorname{supp}\varphi \subset B(\varepsilon,y_0)$. Then $E_+(\varphi(u - (u(y_0) - 1))$ is nonempty and it follows from Theorem I.17 that it is contained in $B(y_0,\varepsilon) \cap A_+$, whence the result.

Remark.1.32. One cannot expect A_+ to be much more than dense (e.g., of full measure, second category, etc.) since $A_+ \cap A_- = d(u)$ may well be empty.

We may also use these results to reformulate the notion of a viscosity solution as follows:

Let $u \in C(\mathcal{O})$ and $y_0 \in \mathcal{O}$. Set

$$D^+u(y_0) = \{a \in R^N: \lim_{y\to y_0} \frac{(u(y)-u(y_0)-a\cdot(y-y_0))^+}{|y-y_0|} = 0\}$$

and

$$D^-u(y_0) = \{a \in R^N: \lim_{y\to y_0} \frac{(u(y)-u(y_0) - a\cdot(y-y_0))^-}{|y-y_0|} = 0\}.$$

where $r^+ = \max(r,0)$, $r^- = -\min(r,0)$. In general, $D^{\pm}u(y_0)$ are empty, but by Corollary I.18 each is nonempty for a dense set of $y_0 \in \mathcal{O}$. The next result is an immediate consequence of the above considerations:

Proposition I.19 Let $u \in C(\ )$. Then:

(i) u is a viscosity solution of $F \leq 0$ if and only if

(1.33) $$F(y,u(y),a) \leq 0 \quad \text{for every } y \in \mathcal{O} \text{ and } a \in D^{+}u(y).$$

(ii) u is a viscosity solution $F \geq 0$ if and only if

(1.34) $$F(y,u(y),a) \geq 0 \quad \text{for every } y \in \mathcal{O} \text{ and } a \in D^{-}u(y).$$

(iii) u is a viscosity solution of $F = 0$ if and only if (1.33) and (1.34) hold.

One can use Proposition I.19 to give another proof of Theorem I.10.

II. Uniqueness for the Dirichlet Problem in $\mathbb{R}^N$.

In paragraph II.1 we treat the simple case

$$u + H(Du) = n(x) \quad \text{in} \quad \mathbb{R}^N. \tag{2.1}$$

After this the general case

$$H(x,u,Du) = 0 \quad \text{in} \quad \mathbb{R}^N, \tag{2.2}$$

which involves technical assumptions, is discussed.

II.1. Equation (2.1).

We consider two problems

$$\begin{cases} \text{(i)} & u + H(Du) = n(x), \\ \text{(ii)} & v + H(Dv) = m(x), \end{cases} \tag{2.3}$$

where

$$H \in C(\mathbb{R}^N), \quad n \in BUC(\mathbb{R}^N), \quad m \in BUC(\mathbb{R}^N). \tag{2.4}$$

The main result concerning (2.3) is:

Theorem II.1. Let (2.4) hold. Let $u,v \in C_b(\mathbb{R}^N)$ be a viscosity subsolution and a viscosity supersolution of (2.3)(i) and (ii) respectively. Then

$$\|(u-v)^+\|_{L^\infty(\mathbb{R}^N)} \le \|(n-m)^+\|_{L^\infty(\mathbb{R}^N)}.^{(\dagger)} \tag{2.5}$$

Remark 2.6. It follows from (2.5) that $n \le m$ implies $u \le v$. It is also an immediate consequence of the theorem that if u,v are viscosity solutions of their respective problems, then $\|(u-v)\|_{L^\infty(\mathbb{R}^N)} \le \|(n-m)\|_{L^\infty(\mathbb{R}^N)}$. In particular, bounded viscosity solutions of (2.1) are unique.

Proof of Theorem II.1. The basic arguments are best illustrated by first running through the proof under the stronger assumption

$$u(x) \to 0 \quad \text{and} \quad v(x) \to 0 \quad \text{as} \quad |x| \to \infty. \tag{2.6}$$

(†) r^+ (r^-) denotes the maximum of r (respectively, $-r$) and 0.

The condition (2.6) is natural if $H(0) = 0$ and $n,m \to 0$ at ∞. After the proof is sketched for the case (2.6), we give the general argument.

<u>Case 1: $u,v \to 0$ as $|x| \to \infty$.</u>

If $u(x) \le v(x)$ everywhere there is nothing to show. Hence assume $u(\bar{x}) - v(\bar{x}) > 0$ for some $\bar{x}$. Let $\varphi \in \mathcal{D}(\mathbb{R}^N)^+$, $0 \le \varphi \le 1$ and $\varphi(0) = 1$. Define

$$M = \max_{\mathbb{R}^N \times \mathbb{R}^N} (\varphi(x-y)(u(x)-v(y)). \tag{2.7}$$

The maximum in (2.7) is assumed and $M > 0$ since $\varphi(\bar{x}-\bar{x})(u(\bar{x})-v(\bar{x})) = u(\bar{x})-v(\bar{x}) > 0$ while $\varphi(x-y)(u(x)-v(y)) \to 0$ as $|x| + |y| \to \infty$ by (2.6) and $\varphi \in \mathcal{D}(\mathbb{R}^N)$. Notice also that for $x \in \mathbb{R}^N$

$$u(x)-v(x) = \varphi(x-x)(u(x)-v(x)) \le M$$

so

$$\|(u-v)^+\|_{L^\infty(\mathbb{R}^N)} \le M. \tag{2.8}$$

Let $M = \varphi(x_0-y_0)(u(x_0)-v(y_0))$, $k_1 = v(y_0)$, $k_2 = u(x_0)$. We then have

$$x_0 \in E_+(\varphi(\cdot - y_0)(u(\cdot) - k_1)) \quad \text{and} \quad y_0 \in E_-(\varphi(x_0 - \cdot)(v(\cdot) - k_2)).$$

It now follows from Theorem I.3 and the assumptions that

$$u(x_0) + H\left(-\frac{(u(x_0)-v(y_0)}{\varphi(x_0-y_0)}(D\varphi)(x_0-y_0)\right) \le n(x_0),$$

$$v(y_0) + H\left(-\frac{(u(x_0)-v(y_0))}{\varphi(x_0-y_0)}(D\varphi)(x_0-y_0)\right) \ge m(y_0)$$

where we used $D_x(\varphi(x-y)) = -D_y(\varphi(x-y))$. Subtracting the above inequalities yields

$$u(x_0)-v(y_0) \le n(x_0) - m(y_0) = n(y_0) - m(y_0) + n(x_0) - n(y_0). \tag{2.9}$$

Choosing φ to be supported in $B(0,\alpha)$ (so $|x_0-y_0| \le \alpha$), (2.9) and $0 \le \varphi \le 1$ imply

$$M \le \|(n-m)^+\|_{L^\infty(\mathbb{R}^N)} + \rho_n(\alpha)$$

where the modulus of continuity ρ_n of n is given by

$$\rho_n(\alpha) = \sup\{|n(x)-n(y)| : |x-y| \le \alpha\}. \tag{2.10}$$

Since $n \in BUC(\mathbb{R}^N)$, we have $\rho_n(\alpha) \to 0$ as $\alpha \to 0$ and the result follows.

Case 2. The general case.

Let $\varphi \in \mathcal{D}(\mathbb{R}^N)^+$ be as above: $0 \le \varphi \le 1$, $\varphi(0) = 1$ and $\operatorname{supp}\varphi \subset B(0,\alpha)$. We are first going to prove, via a truncation argument, that

$$M = \sup_{x,y\in\mathbb{R}^N} \varphi(x-y)(u(x)-v(y)) \le \|(n-m)^+\|_{L^\infty(\mathbb{R}^N)} + \rho_n(\alpha) \tag{2.11}$$

where ρ_n is given by (2.10). The result then follows as before. (The difference between this case and the previous one is that we cannot write "max" in place of "sup" in (2.11).) We may assume $M > 0$.

Let $\varepsilon > 0$,

$$M_\varepsilon = \max_{x,y\in\mathbb{R}^N} \varphi(x-y)(e^{-\varepsilon|x|^2}u(x)-e^{-\varepsilon|y|^2}v(y)),$$

and

$$M_\varepsilon = \varphi(x_\varepsilon-y_\varepsilon)(e^{-\varepsilon|x_\varepsilon|^2}u(x_\varepsilon) - e^{-\varepsilon|y_\varepsilon|^2}v(y_\varepsilon)).$$

Let us first prove that $M_\varepsilon \to M$ as $\varepsilon \downarrow 0$. Since u and v are continuous it is clear that

$$\varliminf_{\varepsilon\downarrow 0} M_\varepsilon \ge M > 0 .$$

Hence, for ε small, $M_\varepsilon \ge M/2$. Moreover, $|x_\varepsilon-y_\varepsilon| \le \alpha$, and one then easily deduces that

$$\sqrt{\varepsilon}\,|x_\varepsilon| \,,\ \sqrt{\varepsilon}\,|y_\varepsilon| \le C \tag{2.12}$$

for some C independent of ε. Now

$$\begin{aligned} M_\varepsilon &= \varphi(x_\varepsilon-y_\varepsilon)(e^{-\varepsilon|x_\varepsilon|^2}u(x_\varepsilon) - e^{-\varepsilon|y_\varepsilon|^2}v(y_\varepsilon)) \\ &\le \varphi(x_\varepsilon-y_\varepsilon)(u(x_\varepsilon) - e^{\varepsilon(|x_\varepsilon|^2-|y_\varepsilon|^2)}v(y_\varepsilon)) \\ &\le \varphi(x_\varepsilon-y_\varepsilon)(u(x_\varepsilon)-v(y_\varepsilon) + (1 - e^{\varepsilon(|x_\varepsilon|^2-|y_\varepsilon|^2)})v(y_\varepsilon)) \\ &\le M + |1 - e^{\varepsilon(|x_\varepsilon|^2-|y_\varepsilon|^2)}|\,|v(y_\varepsilon)|. \end{aligned}$$

However, $|\varepsilon(|x_\varepsilon|^2-|y_\varepsilon|^2)| = \varepsilon|(x_\varepsilon-y_\varepsilon,x_\varepsilon+y_\varepsilon)| \leq \sqrt{\varepsilon}\, 2\alpha C$ by (2.12). Therefore, by the above, $\overline{\lim_{\varepsilon\downarrow 0}} M_\varepsilon \leq M$ and we have $M_\varepsilon \to M$ as $\varepsilon \downarrow 0$.

We next prove (2.11). By

$$\begin{cases} x_\varepsilon \in E_+(\varphi(\cdot - y_\varepsilon)e^{-\varepsilon|\cdot|^2}(u(\cdot) - \psi_1(\cdot))), \psi_1(x) = e^{\varepsilon(|x| - |y_\varepsilon|^2)} v(y_\varepsilon) \\ y_\varepsilon \in E_-(\varphi(x_\varepsilon - \cdot)e^{-\varepsilon|\cdot|^2}(v(\cdot) - \psi_2(\cdot))), \psi_2(y) = e^{\varepsilon(|y|^2 - |x_\varepsilon|^2)} u(x_\varepsilon) \end{cases}$$

and Theorem I.3 we have

$$(2.13) \quad \begin{cases} u(x_\varepsilon) + H(-(u(x_\varepsilon)-k_1) \dfrac{(D\varphi)(x_\varepsilon-y_\varepsilon)}{\varphi(x_\varepsilon-y_\varepsilon)} + 2\varepsilon u(x_\varepsilon)x_\varepsilon) \leq n(x_\varepsilon), \\ k_1 = e^{\varepsilon(|x_\varepsilon|^2 - |y_\varepsilon|^2)} v(y_\varepsilon) \end{cases}$$

and

$$(2.14) \quad \begin{cases} v(y_\varepsilon) + H(-(k_2-v(y_\varepsilon)) \dfrac{(D\varphi)(x_\varepsilon-y_\varepsilon)}{\varphi(x_\varepsilon-y_\varepsilon)} + 2\varepsilon v(y_\varepsilon)y_\varepsilon) \geq m(y_\varepsilon) \\ k_2 = e^{\varepsilon(|y_\varepsilon|^2-|x_\varepsilon|^2)} u(x_\varepsilon). \end{cases}$$

Set

$$(2.15) \quad \begin{cases} \lambda_\varepsilon = -(u(x_\varepsilon) - v(y_\varepsilon)) \dfrac{D\varphi(x_\varepsilon-y_\varepsilon)}{\varphi(x_\varepsilon-y_\varepsilon)}, \\ \delta_\varepsilon = -(1 - \exp(\varepsilon(|x_\varepsilon|^2-|y_\varepsilon|^2)))v(y_\varepsilon) \dfrac{(D\varphi)(x_\varepsilon-y_\varepsilon)}{\varphi(x_\varepsilon-y_\varepsilon)} + 2\varepsilon u(x_\varepsilon)x_\varepsilon, \\ \tilde{\delta}_\varepsilon = (1 - \exp(\varepsilon(|y_\varepsilon|^2-|x_\varepsilon|^2))u(x_\varepsilon) \dfrac{(D\varphi)(x_\varepsilon-y_\varepsilon)}{\varphi(x_\varepsilon-y_\varepsilon)} + 2\varepsilon v(y_\varepsilon)y_\varepsilon. \end{cases}$$

Subtracting the inequalities of (2.13), (2.14) yields

$$M_\varepsilon + H(\gamma_\varepsilon + \delta_\varepsilon) - H(\gamma_\varepsilon + \tilde{\delta}_\varepsilon) \le n(x_\varepsilon) - m(y_\varepsilon) \le \|(n-m)^+\|_{L^\infty(\mathbb{R}^N)} + \rho_n(\alpha).$$

The proof is completed by showing that λ_ε remains bounded as $\varepsilon \downarrow 0$ while δ_ε and $\tilde{\delta}_\varepsilon \to 0$, for then letting $\varepsilon \downarrow 0$ above yields (2.11). Since $M_\varepsilon \ge M/2 > 0$ for ε small, $\varphi(x_\varepsilon - y_\varepsilon)$ is bounded away from zero, proving λ_ε remains bounded. Similarly, $\delta_\varepsilon, \tilde{\delta}_\varepsilon$ tend to zero for $\varepsilon x_\varepsilon$, $\varepsilon y_\varepsilon$ and $\varepsilon(|x_\varepsilon|^2 - |y_\varepsilon|^2)$ tends to zero by (2.12) and the remarks thereafter. This completes the proof.

Remark 2.16. The proof (especially Case 1) is vaguely reminiscent of the proof of uniqueness of entropy solutions of conservation laws in S. N. Kružkov [21].

Remark 2.17. The proofs given used only that n is uniformly continuous and m is continuous. Similarly, we could have used uniform continuity of m and continuity of n. Boundedness of n and m is irrelevant, although the result is not very interesting if $n-m$ is not bounded above. We do not know if the result holds without uniform continuity of at least one of n and m. It is also possible, for example, to replace the boundedness assumptions on u and v by $|u|, |v| \le C(1 + |x|^p)$, $0 < p < 1$ if either H is bounded and uniformly continuous or u and v are Lipschitz continuous. We conjecture that one can take $p = 1$ if u and v are Lipschitz continuous.

II.2. The Equation $H(x,u,Du) = n(x)$.

It will be assumed throughout that $H(x,r,p)$ satisfies

(2.18) For each $R > 0$, H is uniformly continuous on $\mathbb{R}^N \times [-R,R] \times B(0,R)$,

and

(2.19) $\begin{cases} \text{For each } R > 0 \text{ there is a continuous nondecreasing function} \\ \gamma_R : [0,2R] \to \mathbb{R} \text{ such that } \gamma_R(0) = 0 \text{ and} \\ (H(x,r,p) - H(x,s,p)) \ge \gamma_R(r-s) \text{ for } x \in \mathbb{R}^N,\ p \in \mathbb{R}^N,\ -R \le s \le r \le R. \end{cases}$

We will need to restrict the nature of the joint continuity of H. The condition

(2.20) $\lim_{\varepsilon \downarrow 0} \sup\{|H(x,r,p) - H(y,r,p)| : |x-y|(1 + |p|) \le \varepsilon,\ |r| \le R\} = 0$ for all $R > 0$,

and the stronger requirement

$$(2.20^*)\quad \begin{cases} \lim\limits_{\varepsilon\downarrow 0} \sup\{|H(x,r,p) - H(y,r,p)| : |x-y||p| \le R_1,\ |x-y| \le \varepsilon,\ |r| \le R_2\} = 0 \\ \text{for all } R_1, R_2 > 0 . \end{cases}$$

will be used.

We may now state our main result.

<u>Theorem II.2.</u> Let u be a bounded viscosity subsolution of $H(x,u,Du) = 0$ and v be a bounded viscosity supersolution of $H(x,v,Dv) = m(x)$ where $m \in C_b(\mathbb{R}^N)$. Let (2.18), (2.19) hold, $R_0 = \max(\|u\|_{L^\infty(\mathbb{R}^N)}, \|v\|_{L^\infty(\mathbb{R}^N)})$ and $\gamma = \gamma_{R_0}$ as in (2.19). Then:

(i) If (2.20^*) holds we have

$$(2.21)\qquad \|\gamma((u-v)^+)\|_{L^\infty(\mathbb{R}^N)} \le \|m^+\|_{L^\infty(\mathbb{R}^N)} .$$

(ii) If (2.20) holds and $u,v \in BUC(\mathbb{R}^N)$, then (2.21) holds.

(iii) If $u,v \in W^{1,\infty}(\mathbb{R}^N)$, then (2.21) holds.

<u>Remark 2.22.</u> Remarks analogous to (2.6) and (2.17) apply to Theorem II.2.

<u>Remark 2.23.</u> It is not possible to relax the assumptions (2.20), (2.20^*) in an essential way. This can be seen in the linear case $H(x,r,p) = r + b(x) \cdot p$, where (2.20) is equivalent to the Lipschitz continuity of b. See Section V.4 concerning this remark.

<u>Proof of Theorem II.2.</u>

With the notation and assumptions of Step 2 in the proof of Theorems I.2 we have, in the same way,

$$(2.24)\qquad H(x_\varepsilon,u(x_\varepsilon),\gamma_\varepsilon + \delta_\varepsilon) - H(y_\varepsilon,v(y_\varepsilon), \lambda_\varepsilon + \tilde{\delta}_\varepsilon) \le \|m^+\|_{L^\infty(\mathbb{R}^N)} ,$$

where $\lambda_\varepsilon, \delta_\varepsilon, \tilde{\delta}_\varepsilon$ are given by (2.15). Rewrite (2.24) as

$$\begin{aligned} &(H(x_\varepsilon,u(x_\varepsilon),\lambda_\varepsilon+\delta_\varepsilon) - H(x_\varepsilon,v(y_\varepsilon),\lambda_\varepsilon+\delta_\varepsilon)) + (H(x_\varepsilon,v(y_\varepsilon),\lambda_\varepsilon+ \delta_\varepsilon) \\ &- H(y_\varepsilon,v(y_\varepsilon),\lambda_\varepsilon+\delta_\varepsilon)) + (H(y_\varepsilon,v(y_\varepsilon),\lambda_\varepsilon+\delta_\varepsilon) - H(y_\varepsilon,v(y_\varepsilon),\lambda_\varepsilon+\tilde{\delta}_\varepsilon)) \\ &\le \|m^+\|_{L^\infty(\mathbb{R}^N)} . \end{aligned}$$

By (2.19) and $\gamma = \gamma_{R_0}$, this implies

$$\gamma(M_\varepsilon) \le \|m^+\|_{L^\infty(\mathbb{R}^N)} + A_\varepsilon + B_\varepsilon \tag{2.25}$$

where

$$\begin{cases} A_\varepsilon = |H(x_\varepsilon, v(y_\varepsilon), \lambda_\varepsilon+\delta_\varepsilon) - H(y_\varepsilon, v(y_\varepsilon), \lambda_\varepsilon+\delta_\varepsilon)|, \\ B_\varepsilon = |H(y_\varepsilon, v(y_\varepsilon), \lambda_\varepsilon+\delta_\varepsilon) - H(y_\varepsilon, v(y_\varepsilon), \lambda_\varepsilon+\tilde{\delta}_\varepsilon)| \end{cases} \tag{2.26}$$

As we showed before, $\delta_\varepsilon, \tilde{\delta}_\varepsilon \to 0$ while λ_ε remains bounded. Thus, by (2.18), $B_\varepsilon \to 0$ as $\varepsilon \downarrow 0$. We need to estimate A_ε. To this end we reintroduce the support of φ explicity by replacing φ by $\varphi_\alpha(x) = \varphi(x/\alpha)$ where $\varphi \in \mathcal{D}(B(1,0))^+$ $0 \le \varphi \le 1$, $\varphi(0) = 1$, $D\varphi(0) = 0$. Since $\varphi((x_\varepsilon - y_\varepsilon)/\alpha)$ remains bounded away from zero as $\varepsilon \downarrow 0$ we see from (2.15), (2.12) that

$$\limsup_{\varepsilon \downarrow 0}(|\lambda_\varepsilon + \delta_\varepsilon|) < \frac{K}{\alpha}$$

for some K. Since $|x_\varepsilon - y_\varepsilon| \le \alpha$,

$$\limsup_{\varepsilon \downarrow 0} A_\varepsilon \le \sup\{|H(x,r,p) - H(y,r,p)| : |x-y| \le \alpha,\ |r| \le R_0,\ |x-y||p| \le K\}$$

$$= \Lambda(\alpha).$$

Then (2.24) implies $\gamma(M) \le \|M^+\|_{L^\infty(\mathbb{R}^N)} + \Lambda(\alpha)$. If (2.20*) holds, $\Lambda(\alpha) \to 0$ as $\alpha \downarrow 0$ and this proves (i).

To establish case (ii) we will prove that φ can be chosen so that $\lim_{\varepsilon\downarrow 0} |x_\varepsilon - y_\varepsilon| < \alpha\kappa(\alpha)$ for some $\kappa(\cdot)$ satisfying $\kappa(0+) = 0$. Then for ε small, $|x_\varepsilon - y_\varepsilon||\lambda_\varepsilon + \delta_\varepsilon| \le K\kappa(\alpha)$ and the result follows as above. Assume $v \in BUC(\mathbb{R}^N)$ and let ρ_v be the modulus of continuity of v. Recalling the proof of Theorem I.1 we have

$$\sup(u(x)-v(x)) \le \sup \varphi\left(\frac{x-y}{\alpha}\right)(u(x)-v(y)) \le \varliminf_{\varepsilon\downarrow 0} M_\varepsilon \le$$

$$\le \varliminf_{\varepsilon\downarrow 0} \varphi\left(\frac{x_\varepsilon - y_\varepsilon}{\alpha}\right)((u(x_\varepsilon)-v(y_\varepsilon)) + \|v\|_{L^\infty(\mathbb{R}^N)} |\exp(2\alpha c\sqrt{\varepsilon}) - 1|)$$

$$\le \varliminf_{\varepsilon\downarrow 0} \varphi\left(\frac{x_\varepsilon - y_\varepsilon}{\alpha}\right)(u(x_\varepsilon)-v(x_\varepsilon)) + \rho_v(\alpha) + \|v\|_{L^\infty(\mathbb{R}^N)} |\exp(2\alpha c\sqrt{\varepsilon}) - 1|).$$

Without loss of generality we assume $M_0 = \sup(u(x)-v(x)) > 0$. The above inequality then implies, with new constants c_1, c_2 independent of small α and ε ,

$$\varphi\left(\frac{x_\varepsilon - y_\varepsilon}{\alpha}\right) \geq \frac{M_0}{M_0 + \rho_v(\alpha) + c_1\alpha\sqrt{\varepsilon}} \geq 1 - c_2(\rho_v(\alpha) + c_1\alpha\sqrt{\varepsilon})$$

provided that ε is small enough (depending on α). If we choose to be decreasing, radial and $\varphi(x) = 1 - |x|^2$ in $1 \leq 2|x|^2 \leq 2$, the above inequality implies

$$\alpha^2 c_2(\rho_v(\alpha) + c_1\alpha\sqrt{\varepsilon}) \geq |x_\varepsilon - y_\varepsilon|^2$$

when $c_2(\rho_v(\alpha) + c_1\alpha\sqrt{\varepsilon}) \leq 1/2$ and we are done.

For the final case (iii) we use the special case of the following lemma in which w is *Lipschitz continuous:*

<u>Lemma II.3.</u> Let w be continuous on $\mathbb{R}^N$, $\Phi \in C^1(\mathbb{R}^N)$ and $x_0 \in E^+(w\Phi)$. Set

$$\rho_w(\lambda) = \max\{|w(x_0) - w(x)| : |x_0 - x| \leq \lambda\},$$

and

$$\rho_{D\Phi(\lambda)} = \max\{|D\Phi(x_0) - D\Phi(x)| : |x_0 - x| \leq \lambda\} .$$

Then for $\lambda > 0$ with $w(x_0) > \rho_w(\lambda)$

$$w(x_0)\frac{|D\Phi(x_0)|}{\Phi(x_0)} \leq \frac{\rho_w(\lambda)}{\lambda}\frac{w(x_0)}{w(x_0) - \rho_w(\lambda)} + w(x_0)\rho_{D\Phi}(\lambda) .$$

In particular, if $Dw \in L^\infty(B(x_0,R))$ for some $R > 0$, then

$$w(x_0)\frac{|D\Phi(x_0)|}{\Phi(x_0)} \leq \|Dw\|_{L^\infty(B(x_0,R))} .$$

We first complete the proof of the theorem and then prove the lemma. Recall (2.25), (2.26) and that

$$\lambda_\varepsilon + \delta_\varepsilon = -(u(x_\varepsilon) - \psi_1(x_\varepsilon))\frac{D\Phi(x_\varepsilon)}{\Phi(x_\varepsilon)} + D\psi_1(x_\varepsilon)$$

where $\Phi(x) = e^{-\varepsilon|x|^2}\varphi((x-y_\varepsilon)/\alpha)$, $\psi_1 = e^{\varepsilon(|x|^2 - |y_\varepsilon|^2)}$,

$\sqrt{\varepsilon}(|x_\varepsilon| + |y_\varepsilon|) \leq c$ and $x_\varepsilon \in E_+((u-\psi_1)\Phi)$. It follows from Lemma II.3 that the first

term on the right above is bounded by

$$\|Du\|_{L^\infty(\mathbb{R}^N)} + \|2\varepsilon e^{\varepsilon(|x|^2-|y_\varepsilon|^2)}\|_{L^\infty(B(1,x_\varepsilon))}$$

which is bounded independent of ε,α . The term $D\psi_1(x_\varepsilon) \to 0$ as $\varepsilon \downarrow 0$ uniformly in α . Thus (2.18) implies $\lim_{\alpha\downarrow 0} A_\varepsilon = 0$ uniformly in ε , and the proof is complete.

<u>Proof of Lemma II.3.</u>

Let $\lambda > 0$, $\rho_w(\lambda) < w(x_0)$ and set $\hat{x} = |D\Phi(x_0)|^{-1}D\Phi(x_0)$. Set

$$\lambda\partial^2\Phi(\lambda) = \Phi(x_0+\lambda\hat{x}) - (\Phi(x_0) + \lambda D\Phi(x_0)\hat{x})$$

$$= \Phi(x_0+\lambda\hat{x}) - (\Phi(x_0) + \lambda|D\Phi(x_0)|) .$$

Then

$$w(x_0 + \lambda\hat{x})\Phi(x_0+\lambda\hat{x}) \le w(x_0)\Phi(x_0)$$

implies

$$w(x_0+\lambda\hat{x})(\Phi(x_0)+\lambda|D\Phi(x_0)| + \lambda\partial^2\Phi(\lambda)) \le w(x_0)\Phi(x_0)$$

or

$$w(x_0)\frac{|D\Phi(x_0)|}{\Phi(x_0)} \le \frac{w(x_0)}{w(x_0+\lambda\hat{x})}\frac{(w(x_0)-w(x_0+\lambda\hat{x}))}{\lambda} + w(x_0)\partial^2\Phi(\lambda) ,$$

where the manipulations are justified by $w(x_0+\lambda\hat{x}) \ge w(x_0) - \rho_w(\lambda) > 0$. The result now follows from $w(x_0) - w(x_0+\lambda\hat{x})| \le \rho_w(\lambda)$, $w(x_0+\lambda\hat{x}) \ge w(x_0) - \rho_w(\lambda)$, $|\partial^2\Phi(\lambda)| \le \rho_{D\Phi}(\lambda)$. The final assertion follows from the relations $\rho_w(\lambda)/\lambda \le \|Dw\|_{L^\infty(B(x_0,R))}$ for $\lambda \le R$, $\rho_{D\Phi}(0+) = 0$, and letting $\lambda \downarrow 0$ in the inequality.

III. Uniqueness for the Dirichlet Problem in Ω.

In this section we turn to the uniqueness question for

$$(3.1)\qquad \begin{cases} H(x,u,Du) = 0 & \text{in} \quad \Omega, \\ u(x) = z(x) & \text{on} \quad \partial\Omega, \end{cases}$$

in the case where Ω is an open subset of $\mathbb{R}^N$ and $\partial\Omega \neq \phi$. In this section the restrictions (2.18)-(2.20*) on H are to be understood by replacing $\mathbb{R}^N$ by Ω. The main result is:

Theorem III.1. Let $u,v \in C_b(\bar{\Omega})$ and (2.18), (2.19) hold. Let u,v be viscosity solutions of $H(x,u,Du) = 0$ and $H(x,v,Dv) = m$ in Ω where $m \in C_b(\bar{\Omega})$. Let $R_0 = \max(\|u\|_{L^\infty(\Omega)}, \|v\|_{L^\infty(\Omega)})$ and $\gamma = \gamma_{R_0}$ from (2.19). Then

(i) If (2.20*) holds and $u|_{\partial\Omega}$ or $v|_{\partial\Omega}$ is uniformly continuous and

$$\lim_{\substack{x\in\Omega \\ x \to x_0}} (|u(x) - u(x_0)| + |v(x) - v(x_0)|) = 0$$

uniformly for $x_0 \in \partial\Omega$, then

$$(3.2)\qquad \|\gamma(u-v)^+)\|_{L^\infty(\Omega)} \leq \max(\|m^+\|_{L^\infty(\Omega)}, \|\gamma((u-v)^+)\|_{L^\infty(\partial\Omega)}).$$

(ii) If (2.18), (2.19), (2.20) hold and $u,v \in BUC(\bar{\Omega})$ Then (3.2) holds.

(iii) If (2.18), (2.19) hold and $u,v \in W^{1,\infty}(\Omega)$, then (3.2) holds.

Remark 3.3. *Remarks analogous to (2.6), (2.17) are valid here.*

Proof of Theorem III.1. We give the proof only in the case when Ω is bounded. The general case follows from a combination of the arguments given below and in the proof of Theorem II.

Without loss of generality we may assume $\|(u-v)^+\|_{L^\infty(\Omega)} > \|(u-v)^+\|_{L^\infty(\partial\Omega)}$. Then (3.2) reduces to

$$\|\gamma((u-v)^+)\|_{L^\infty(\Omega)} < \|m^+\|_{L^\infty(\Omega)} .$$

Let $\varphi_\alpha(x) = \varphi(x/\alpha)$ as in the end of the proof of Theorem II.2 and

$$M_\alpha = \sup_{x,y\in\bar\Omega} \varphi_\alpha(x-y)(u(x)-v(y)).$$

Now $u,v \in C_b(\bar\Omega) = BUC(\bar\Omega)$ since $\bar\Omega$ is compact. With $M_0 = \|(u-v)^+\|_{L^\infty(\Omega)}$ we therefore clearly have

$$M_0 \le M_\alpha \le \varphi_\alpha(x_\alpha-y_\alpha)(M_0+\rho_v(\alpha)) \tag{3.4}$$

where ρ_v is the modulus of continuity of v and $x_\alpha,y_\alpha \in \bar\Omega$, $\varphi_\alpha(x_\alpha-y_\alpha)(u(x_\alpha)-v(y_\alpha)) = M_\alpha$. From (3.4) and the choice of φ_α we deduce $|x_\alpha-y_\alpha| \le \alpha\delta(\alpha)$ where $\delta(0+) = 0$ as in the proof of Theoerm II.2(ii). Finally, as $\alpha \downarrow 0$ all limit points of x_α,y_α lie in $E_+((u-v)) \subset \Omega$. Therefore, there is a compact $K \subset \Omega$ such that $x_\alpha,y_\alpha \in K$ for α small. It follows that $\varphi_\alpha(\cdot - y_\alpha), \varphi_\alpha(x_\alpha - \cdot) \in (\Omega)^+$ for small α. From the assumptions we conclude:

$$H(x_\alpha,u(x_\alpha), -(u(x_\alpha) - v(y_\alpha)) \frac{(D\varphi_\alpha)(x_\alpha-y_\alpha)}{\varphi_\alpha(x_\alpha-y_\alpha)}) \le 0 ,$$

$$H(y_\alpha,v(y_\alpha), -(u(x_\alpha) - v(y_\alpha)) \frac{(D\varphi_\alpha)(x_\alpha-y_\alpha)}{\varphi_\alpha(x_\alpha-y_\alpha)}) \ge m(y_\alpha)$$

which implies (recall the proof of Theorem II.2)

$$\gamma(M_\alpha) \le \|m^-\|_{L^\infty(\Omega)} + \sup_{\substack{|x-y|\le\alpha\delta(\alpha)\\ |r| \le R_0\\ |p| \le c/\alpha}} |H(x,r,p) - H(y,r,p)|.$$

for some c. Moreover, if $Du,Dv \in W^\infty(\Omega)$ we may replace $|p| \le c/\alpha$ by $|p| \le c$. The argument concludes in the usual way.

Remark 3.5. The condition (2.19) can be weakened to $H(x,r,p) - H(x,s,p) \geq \gamma_{R,\delta}(r-s)$ for $-R \leq s \leq r \leq R$, $p \in \mathbb{R}^N$ and $x \in \Omega_\delta = \{x: \text{distance } (x,\partial\Omega) > \delta\}$ with the conclusion being $u \leq v$ if $m \geq 0$ and $u \leq v$ on $\partial\Omega$.

All the above results require that $H(x,r,p)$ be strictly increasing in r. Moreover uniqueness fails without some monotonicity in this sense. An extreme example is $H \equiv 0$. We treat one case without strict monotonicity in r via an adaptation of a device of S. N. Kružkov [18].

For simplicity consider the example

$$H(Du) = n(x) \quad \text{in} \quad \Omega . \tag{3.6}$$

where we assume

$$\begin{cases} H(0) = 0 , \quad H \text{ is convex, continuous and } H \geq 0 , \\ n \in C(\bar{\Omega}) , \quad n > 0 \text{ in } \bar{\Omega}, \ \Omega \text{ is bounded.} \end{cases} \tag{3.7}$$

Proposition III.2. Let (3.7) hold and $u,v \in C(\bar{\Omega})$ be viscosity sub- and supersolutions, respectively, of (3.6). Then

$$\|(u-v)^+\|_{L^\infty(\partial\Omega)} \leq \|(u-v)^+\|_{L^\infty(\partial\Omega)} .$$

Proof Let $\Psi \in C^\infty(\mathbb{R})$ satisfy $\Psi' > 0$, $\Psi'' > 0$ everywhere and $\Psi(\mathbb{R}) = \mathbb{R}$. Let $\Phi = \Psi^{-1}$. By Corollary I.8, $\tilde{u} = \Phi(u)$, $\tilde{v} = \Phi(v)$ are viscosity sub- and supersolutions, respectively, of

$$\frac{1}{\Psi'(w)} H(\Psi'(w)Dw) = \frac{1}{\Psi'(w)} n(x) \quad \text{in} \quad \Omega . \tag{3.8}$$

The Hamiltonian

$$\tilde{H}(x,r,p) = \frac{1}{\Psi'(r)} H(\Psi'(r)p) - \frac{1}{\Psi'(r)} n(x)$$

is locally Lipschitz in r and a computation yields

$$\frac{\partial}{\partial r} \tilde{H} = \frac{\Psi''(r)}{(\Psi'(r))^2} [(DH)(\Psi'(r)p)\Psi''(r)p - H(\Psi'(r)p)] + \frac{\Psi''(r)}{(\Psi'(r))^2} n(x) .$$

Since H is convex $DH(q) \cdot q - H(q) \geq -H(0) = 0$ and we deduce

$$\frac{\partial \tilde{H}}{\partial r} \geq \frac{\Psi''(r)}{(\Psi'(r))^2} \, n(x).$$

Therefore $\tilde{H}$ satisfies the conditions of Theorem III.1(i) and we obtain

$$\|(\Phi(u) - \Phi(v))^+\|_{L^\infty(\Omega)} \leq \|(\Phi(u) - \Phi(v))^+\|_{L^\infty(\partial\Omega)} .$$

Since Ψ can be replaced by $\Psi_\theta(r) = \theta\Psi(r) + (1-\theta)r$ for any $\theta \in]0,1]$, we deduce

$$\|(\Phi_\theta(u) - \Phi_\theta(v)^+\|_{L^\infty(\Omega)} \leq \|(\Phi_\theta(u) - \Phi_\theta(v))^+\|_{L^\infty(\partial\Omega)}$$

where $\Phi_\theta = (\Psi_\theta)^{-1}$. To conclude, we observe that $\Phi_\theta(r) \to r$ locally uniformly as $\theta \to 0^+$.

<u>Remark 3.8.</u> It is worth noting that uniqueness of (viscosity) solutions of (3.6) may fail if we assume only:

$$(3.9) \qquad \begin{cases} H(0) = 0 \,, \quad H \text{ is convex, continuous and } H \geq 0 \\ n \in C(\bar{\Omega}), \quad n \geq 0 \text{ in } \bar{\Omega} \,, \quad \Omega \text{ is bounded.} \end{cases}$$

Actually it is enough for n to vanish at <u>one</u> point to imply in general the nonuniqueness as it is shown in the following example: let $\Omega = [-1,+1]$, $H(p) = |p|^2$, $n(x) = x^4$. Clearly $\bar{u}(x) = \frac{1}{3} - \frac{1}{3}|x|^3$ is a C^1 solution of:

$$|\bar{u}'|^2 = |x|^4 \text{ in } \Omega \,, \quad \bar{u} = 0 \quad \text{on} \quad \partial\Omega \,.$$

On the other hand, if we let $u(x) = \frac{1}{3} - \frac{1}{3}|x|^3$ for $|x| \geq t_0$ and $u(x) = \frac{1}{3}|x|^3$ for $|x| \leq t_0$, where $t_0 = 2^{-1/3}$, u is a solution of the same equation which is also in C^1 except at $\pm t_0$ where the discontinuity of u' is such that u is still a viscosity solution. Therefore in this example we have two different viscosity solutions.

As remarked in the introduction, all the above uniqueness results are new. No uniqueness criteria (even for generalized solutions in $W^{1,\infty}(\Omega)$) are known except in the case of a convex Hamiltonian. In the convex case, A. Douglis [10] and S. N. Kružkov [18] have introduced the class of semi-concave functions, that is functions u such that

$\frac{\partial^2 u}{\partial \chi^2} \leq C_\delta$ in $\mathcal{D}'(\Omega_\delta)$ for all $\delta > 0$ and for all $\chi : |\chi| = 1$ with Ω_δ defined in (3.5) and χ denoting an arbitrary direction. Uniqueness in this class is proved by the above authors. P. L. Lions [22] (see also [23]) extends these results to the class of functions satisfying:

$$\Delta u \leq C_\delta \quad \text{in} \quad \mathcal{D}'(\Omega_\delta) \quad \text{for all} \quad \delta > 0 .$$

All these results require convex Hamiltonians and some degree of regularity of the solutions.

To conclude this section, we observe that in the convex case any Lipschitz subsolution is a viscosity subsolution and any Lipschitz, semi-concave supersolution is a viscosity supersolution. (This implies, by the way, that the uniqueness results of Douglis and Kružkov are completely contained in ours).

<u>Proposition III.3:</u> Let $H(x,r,p)$ be a continuous Hamiltonian, convex in p.

(i) Let $u \in W^{1,\infty}_{loc}(\Omega)$ satisfy: $H(x,u,Du) \leq 0$ in Ω then u is a viscosity subsolution of $H(x,v,Dv) = 0$.

(ii) Let u be a *locally bounded semiconcave function satisfying:*

$$H(x,u,Du) \geq 0 \quad \text{in} \quad \Omega ,$$

then u is a viscosity supersolution of $H(x,v,Dv) = 0$.

<u>Proof of Proposition III.3.</u>

(i) We first remark that if u is a locally Lipschitz subsolution of:

$$H(x,u,Du) \leq 0 \quad \text{in} \quad \Omega ,$$

then an easy argument shows that we have:

$$H(x,u^\varepsilon,Du^\varepsilon) \leq f_\varepsilon(x) \quad \text{in} \quad \Omega_\varepsilon$$

where $f_\varepsilon \to 0$ uniformly on compact sets of Ω and $u_\varepsilon = u * p_\varepsilon$ with $p_\varepsilon = \frac{1}{\varepsilon^N} p(\frac{\cdot}{\varepsilon})$,

$p \in \mathcal{D}_+(\mathbb{R}^N)$, supp $p \subset B_1$, $\|p\|_{L^1} = 1$ (Observe that $H(Du_\varepsilon) \leq H(Du) * p_\varepsilon$ if H is convex).

Now since u^ε is C^∞, u^ε is obviously a viscosity subsolution of the equation: $H(x,v,Dv) = f_\varepsilon(x)$ in Ω_{ε_0} (for any $0 < \varepsilon \leq \varepsilon_0$). Thus we conclude by a simple application of Theorem I.2.

(ii) Let u be a locally bounded semi-concave function satisfying: $H(x,u,Du) \geq 0$ on Ω. Without loss of generality (restricting if necessary our attention to each Ω_δ, and making a translation) we may assume: $u \in W^{1,\infty}(\Omega)$, u is concave on Ω or more precisely: $\frac{\partial^2 u}{\partial \chi^2} \leq 0$ in $\mathcal{D}'(\Omega)$ $\forall \chi : |\chi| = 1$ (This implies that u is concave on every convex subset of $\bar{\Omega}$).

Now let φ, k be such that $E_-(\varphi(u-k)) \neq \phi$, $\varphi \in \mathcal{D}_+(\Omega)$, $k \in \mathbb{R}$ and let $x_0 \in E_-(\varphi(u-k))$. Obviously, there exists $\rho > 0$ small enough such that on $B(x_0,\rho)$ we have:

$$u(x_0) \geq k + \frac{\varphi(x_0)}{\varphi(x)}(u(x_0)-k) = u(x_0) - \frac{D\varphi(x_0)}{\varphi(x_0)}(u(x_0)-k)\cdot(x-x_0) + |x-x_0|\varepsilon(x)$$

where $\varepsilon(x) \to 0$ as $|x-x_0| \to 0$. Since u is concave on $B(x_0,\rho)$, this inequality implies that u is differentiable at x_0 and $Du(x_0) = -\frac{D\varphi(x_0)}{\varphi(x_0)}(u(x_0) - k)$. To conclude we just have to prove that $H(x_0,u(x_0),Du(x_0)) \geq 0$. But by assumption $\exists\, x_n \in \Omega$, $x_n \xrightarrow[n\to\infty]{} x_0$, u is differentiable at x_n and

$$H(x_n,u(x_n),Du(x_n)) \geq 0 .$$

And since u is concave, we have $Du(x_n) \to Du(x_0)$ (all limit points of $Du(x_n)$ are superdifferentials of u at x_0 and therefore reduce to $Du(x_0)$).

IV. Existence of Viscosity Solutions of the Dirichlet Problem.

In this section we establish that the most common method of obtaining generalized solutions of HJ equations actually provides viscosity solutions. This is done in paragraph IV.1 and roughly means that we could take all known existence theorems and generalize (using Theorem I.1 in the process) and restate them as results concerning viscosity solutions. Of course we will not do this - we refer the reader instead to [22] for a complete treatment of general results of this sort and references to the earlier literature. However, it seems worthwhile to illustrate the situation by giving very general new results for a simple model problem, which we do in paragraph IV.2.

IV.1 The Method of Vanishing Viscosity and Viscosity Solutions of HJ Equations.

The vanishing viscosity method for obtaining solutions of

$$H(x,u,Du) = 0 \quad \text{in} \quad \Omega\ , \quad u = z \quad \text{on} \quad \partial\Omega \tag{4.1}$$

consists of approximating the problem by ones of the form

$$(4.1)_\varepsilon \qquad \begin{array}{ll} \text{(a)} & -\varepsilon\Delta u_\varepsilon + H_\varepsilon(x,u_\varepsilon,Du_\varepsilon) = 0 \quad \text{in} \quad \Omega\ , \\ \text{(b)} & u_\varepsilon = z_\varepsilon \quad \text{on} \quad \partial\Omega \end{array}$$

where $\varepsilon > 0$, H_ε, z_ε are adequately smooth and converge locally uniformly to H,z respectively. One attempts to prove $(4.1)_\varepsilon$ is solvable for $\varepsilon > 0$, and to obtain precompactness of the family $\{u_\varepsilon : 0 < \varepsilon < 1\}$ in $C(\Omega)$ (or $C(\bar{\Omega})$)). Typically this is done by obtaining (perhaps local) estimates on u_ε and Du_ε in L^∞ . See [18] and IV.2 below in this regard. We prove:

<u>Proposition IV.1.</u> Let $u_\varepsilon \in C^2(\Omega)$ be a solution of $(4.1)_\varepsilon$(a) where $H_\varepsilon \to H$ as $\varepsilon \downarrow 0$ in $C(\Omega \times R \times R^n)$. Assume $\varepsilon_n \downarrow 0$ and $u_{\varepsilon_n} \to u$ in $C(\Omega)$ and $n \to \infty$. Then u is a viscosity solution of $H(x,u,Du) = 0$. If also $u_\varepsilon = z_\varepsilon$ on $\partial\Omega$, $z_\varepsilon \to z$ in $C(\partial\Omega)$ and $u_{\varepsilon_n} \to u$ in $C(\bar{\Omega})$ then $u|_{\partial\Omega} = z$.

Proof. Let $u_{\varepsilon_n} \to u$ in $C(\Omega)$ as in the assumptions. Fix $\varphi \in \mathcal{D}(\Omega)^+$, $k \in R^N$ and assume $E_+(\varphi(u-k)) \neq \phi$. Then for large n there exists $x_n \in E_+(\varphi(u_{\varepsilon_n} - k))$ and, passing to a subsequence if necessary, we may assume $x_n \to x \in E_+(\varphi(u-k))$. By a simple computation we have on $\operatorname{supp}\varphi$

$$0 = \frac{1}{\varphi}\left(\varphi(-\varepsilon\Delta u_\varepsilon + H(x,u_\varepsilon,Du_\varepsilon))\right)$$

$$= -\varepsilon\frac{1}{\varphi}\Delta(\varphi(u_\varepsilon-k)) + \varepsilon(u_\varepsilon-k)\frac{\Delta\varphi}{\varphi} + 2\varepsilon\frac{D\varphi\cdot D(\varphi(u_\varepsilon-k))}{\varphi^2}$$

$$-2\varepsilon\frac{(u_\varepsilon-k)}{\varphi^2}|D\varphi|^2 + H(x,u_\varepsilon,\frac{1}{\varphi}D(\varphi(u_\varepsilon-k)) - \frac{u_\varepsilon-k}{\varphi}D\varphi).$$

Evaluating this identity at $\varepsilon = \varepsilon_n$, $x = x_n$ and using $(\Delta(\varphi(u_\varepsilon-k)))(x_n) \leq 0$, $(D\varphi(u_\varepsilon-k))(x_n) = 0$ (because $x_n \in E_+(\varphi(u_{\varepsilon_n}-k))$ we conclude

$$\varepsilon_n(u_{\varepsilon_n}(x_n)-k)\frac{\Delta\varphi(x_n)}{\varphi(x_n)} - 2\varepsilon_n(u_{\varepsilon_n}(x_n)-k)\frac{|D\varphi(x_n)|^2}{\varphi(x_n)^2}$$

$$+ H(x_n,u_{\varepsilon_n}(x_n), -(u_{\varepsilon_n}(x_n)-k)\frac{D\varphi(x_n)}{\varphi(x_n)}) \leq 0 .$$

Since $x_n \to x \in E_+(\varphi(u-k))$ we find, letting $n \to \infty$, $H(x,u(x),-(u(x)-k)\frac{D\varphi(x)}{\varphi(x)}) \leq 0$. Thus u is a viscosity subsolution. Similarly, it is a viscosity supersolution and the result follows.

Remark 4.2. We could replace $u_\varepsilon \in C^2(\Omega)$ above by $u_\varepsilon \in W^{2,p}_{loc}(\Omega)$, $p > N$, via Bony's maximum principle [5].

Remark 4.3. If we obtain a viscosity solution of (4.1) in this way and one of our uniqueness results applies, it follows that u_ε converges to this unique solution as $\varepsilon \downarrow 0$. This is known in some particular cases via arguments using considerations of control theory or differential games (W. H. Fleming [14,15], A. Friedman [16]).

Remark 4.4. This result also shows that the optimal cost function $\bar{u}$ of the control problem associated with (4.1) (or the value function in the case of differential games (see S. H. Benton [4], W. H. Fleming [13,14,15]) is indeed a (or the) viscosity solution.

Indeed, in these contexts it is easy to show u_ε converges to $\bar{u}$, and the Theorem applies.

IV.2. A Model Equation.

We will assume

$$(4.4)\qquad \begin{cases} \text{(i)} & H \in C(\mathbb{R}^N), \\ \text{(ii)} & \beta:\mathbb{R} \to \mathbb{R} \text{ is an increasing homeomorphism of } \mathbb{R} \text{ onto } \mathbb{R}, \\ \text{(iii)} & n \in BUC(\mathbb{R}^N). \end{cases}$$

and consider the model problem

$$(4.5)\qquad \beta(u) + H(Du) = n \quad \text{in} \quad \mathbb{R}^N .$$

It simplifies the discussion to follow to assume

$$(4.6)\qquad H(0) = 0 , \quad \beta(0) = 0 ,$$

which amounts to changing n by a constant. We will consider solutions of approximate problems of the form

$$(4.7)\qquad -\varepsilon\Delta u_\varepsilon + \beta_\varepsilon(u_\varepsilon) + \varepsilon u_\varepsilon + H_\varepsilon(Du_\varepsilon) = n_\varepsilon$$

under assumptions given later. Before doing so we obtain the key estimates we need. This also motivates Proposition IV.3 concerning (4.5).

<u>Lemma IV.2.</u> Let $F \in C(\mathbb{R}^N)$, $F(0) = 0$, and γ be an increasing homeomorphism of $\mathbb{R}$, $\gamma(0) = 0$. Assume $v,\hat{v} \in C^2(\mathbb{R}^N) \quad L^\infty(\mathbb{R}^N)$, $F(Dv)$, $F(D\hat{v}) \in L^\infty(\mathbb{R}^N)$ and

$$(4.8)\qquad \begin{cases} \text{(a)} & -\varepsilon\Delta v + \gamma(v) + F(Dv) = m \in C_b(\mathbb{R}^N) \\ \text{(b)} & -\varepsilon\Delta \hat{v} + \gamma(\hat{v}) + F(D\hat{v}) = \hat{m} \in C_b(\mathbb{R}^N). \end{cases}$$

Then for $\nu \in \{+,-\}$

$$(4.9)\qquad \|\gamma(v)^\nu\|_{L^\infty(\mathbb{R}^N)} \le \|m^\nu\|_{L^\infty(\mathbb{R}^N)} , \quad \|\gamma(\hat{v})^\nu\|_{L^\infty(\mathbb{R}^N)} \le \|\hat{m}^\nu\|_{L^\infty(\mathbb{R}^N)}$$

and

$$(4.10)\qquad \|(v-\hat{v})^+\|_{L^\infty(\mathbb{R}^N)} \le \sup\{|\gamma^{-1}(s+\|(m-\hat{m})^+\|_{L^\infty(\mathbb{R}^N)}) - \gamma^{-1}(s))| : |s| \le \|\hat{m}\|_{L^\infty(\mathbb{R}^N)}\}.$$

Sketch of Proof. If $x \in E_+(v)$ then $\Delta v(x) \le 0$ and $F(Dv(x)) = F(0) = 0$. Hence, from (4.8), $\gamma(v(x)) \le m(x)$, and we would have (4.9) with $\nu = +$. If $E_+(v) = \phi$ but $v > 0$ somewhere, one chooses $x_\lambda \in E_+(e^{-\lambda|x|^2} v)$ makes the associated computation and uses $\lambda|x_\lambda|^2 \le C$ to let $\lambda \downarrow 0$ and reach the same conclusion. For this we need to observe that $Dv \in L^\infty(\mathbb{R}^N)$ because $v \in L^\infty(\mathbb{R}^N)$ and $-\varepsilon\Delta v = m - F(Dv) - \gamma(v) \in L^\infty(\mathbb{R}^N)$ by assumption. To understand (4.10), let $x \in E_+(v-\hat{v})$. Forming the difference of (4.8) (a) and (b) and using $\Delta(v-\hat{v})(x) \le 0$, $F(Dv) = F(D\hat{v})$ at x one finds $\gamma(v(x)) - \gamma(\hat{v}(x)) \le m(x) - \hat{m}(x)$. Writing $v(x) = \hat{v}(x) + \|(v-\hat{v})^+\|_{L^\infty(R^N)}$ we have

$$\gamma(\mu+r)-\gamma(\mu) \le \|(m-\hat{m})^+\|_{L^\infty(\mathbb{R}^N)}, \quad \mu = \hat{v}(x), \quad r = \|(v-\hat{v})\|_{L^\infty(R^N)}.$$

But then

$$r \le \gamma^{-1}(\gamma(\mu) + \|(m-\hat{m})\|_{L^\infty(\mathbb{R}^N)}) - \gamma^{-1}(\gamma(\mu))$$

and we have (4.10). If $E_+(v-\hat{v}) = \phi$ but $v-\hat{v} > 0$ somewhere, approximate by $x_\lambda \in E_+(e^{\lambda|\cdot|^2}(v-\hat{v}))$ and let $\lambda \downarrow 0$. This completes the discussion of Lemma IV.2.

The main result concerning (4.5) is:

Proposition IV.3. Let (4.4), (4.6) hold. Then (4.5) has a unique viscosity solution $u \in C_b(\mathbb{R}^N)$. Moreover,

(4.11) $$\|\beta(u)^\nu\|_{L^\infty(\mathbb{R}^N)} \le \|n^\nu\|_{L^\infty(\mathbb{R}^N)}, \quad \nu \in \{+,-\},$$

(4.12) If $m \in BUC(\mathbb{R}^N)$ and v is the viscosity solution of $\beta(v) + H(Dv) = m$, then

$$\|(u-v)^+\|_{L^\infty(\mathbb{R}^N)} \le \sup\{|\beta^{-1}(s+\|(n-m)^+\|_{L^\infty(\mathbb{R}^N)}) - \beta^{-1}(s)| : |s| \le \|m\|_{L^\infty(\mathbb{R}^N)}\}.$$

(4.13) If ρ_u, ρ_n are the moduli of continuity of u, n, respectively, then

$$\rho_u(r) \le \sup\{\beta^{-1}(s+\rho_n(r)) - \beta^{-1}(s) : |s| \le \|m\|_{L^\infty(\mathbb{R}^N)}\}.$$

Sketch of Proof of Proposition IV.3. The uniqueness of viscosity solutions of (4.5) follows from THeorem II.2. The Hamiltonian $\bar{H}(x,r,p) = \beta(r) + H(p) - n(x)$ clearly satisfies (2.18). For (2.19) we note that

$$\bar{H}(x,r,p) - \bar{H}(x,s,p) = \beta(r) - \beta(s) \ge \gamma_R(r-s), \quad -R \le s \le r \le R$$

with $\gamma_R(\tau) = \inf\{\beta(s+\tau) - \beta(s) : |s| \le R\}$ for $\tau \ge 0$. Finally, (2.20*) reduces to the uniform continuity of n.

For the existence, let $\beta_\varepsilon, H_\varepsilon, n_\varepsilon \in C^\infty$ be approximations of β, H, n such that

$$(4.14)\quad \begin{cases} \beta_\varepsilon,\ \beta_\varepsilon' \in L^\infty(R),\ \beta_\varepsilon' \ge 0,\ \beta_\varepsilon(0) = 0\ ,\ \beta_\varepsilon \to \beta \text{ in } C(\mathbb{R}) \text{ as } \varepsilon \downarrow 0\ , \\ n_\varepsilon \in BUC(\mathbb{R}^N) \text{ and } n_\varepsilon \to n \text{ uniformly as } \varepsilon \downarrow 0, \\ H_\varepsilon \in L^\infty(\mathbb{R}^N),\ H_\varepsilon(0) = 0\ ,\ \text{and } H_\varepsilon \to H \text{ in } C(\mathbb{R}^N) \text{ as } \varepsilon \downarrow 0\ . \end{cases}$$

It is then nearly trivial that

$$(4.15)\qquad -\varepsilon\Delta u_\varepsilon + \beta_\varepsilon(u_\varepsilon) + \varepsilon u_\varepsilon + H_\varepsilon(Du_\varepsilon) = n_\varepsilon$$

has a solution $u_\varepsilon \in C^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. One can simply solve the associated truncated problem in $B(0,R)$ for $u_{\varepsilon R}$ subject to $u_{\varepsilon R} = 0$ on $|x| = R$. Then $|\beta_\varepsilon(u_{\varepsilon R}) + \varepsilon u_{\varepsilon R}| \le \|n_\varepsilon\|_{L^\infty(B(0,R))}$ follows as in Lemma IV.2. Using $H_\varepsilon \in L^\infty$ and interior estimates we conclude $-\varepsilon\Delta u_{\varepsilon R}$ is bounded in $L^\infty(B(0,R))$ as $R \to \infty$ and by compactness there is a sequence $R_n \to \infty$ and $u_\varepsilon \in C_b^1(\mathbb{R}^N)$, $\Delta u_\varepsilon \in L^\infty(\mathbb{R}^N)$, such that $u_{R_n} \to u_\varepsilon$ boundedly in $C^1_{loc}(\mathbb{R}^N)$ while $\Delta u_{\varepsilon R_n} \to \Delta u_\varepsilon$ weakly in $L^2_{loc}(\mathbb{R}^N)$. Then (4.15) implies $u_\varepsilon \in C^\infty(\mathbb{R}^N)$.

Using Lemma IV.2 we conclude

$$(4.16)\qquad \|(\beta_\varepsilon(u_\varepsilon) + \varepsilon u_\varepsilon)^\nu\|_{L^\infty(\mathbb{R}^N)} \le \|n_\varepsilon^\nu\|_{L^\infty(\mathbb{R}^N)}\ .$$

Since $\beta_\varepsilon \to \beta$ locally uniformly and $\beta(\mathbb{R}) = \mathbb{R}$, (4.4) implies u_ε is bounded in $L^\infty(\mathbb{R}^N)$. Moreover, $u_\varepsilon(\cdot+y)$ solves (4.15) with n_ε replaced by $n_\varepsilon(\cdot+y)$. By Lemma IV.2) we therefore have

$$(4.17)\quad |u_\varepsilon(x+y) - u_\varepsilon(x)| \le \sup\{|(\beta_\varepsilon+\varepsilon I)^{-1}(s+\rho_{n_\varepsilon}(|y|)) - (\beta_\varepsilon+\varepsilon I)^{-1}(s)| : |s| \le \|n_\varepsilon\|_{L^\infty(R^N)}\}$$

where ρ_{n_ε} is the modulus of continuity of n_ε. It is easy to choose n_ε so that $\rho_{n_\varepsilon} \le \rho_n$, and we assume we have done so. Moreover, since $\beta_\varepsilon + \varepsilon I \to \beta$ locally uniformly, $(\beta_\varepsilon + \varepsilon I)^{-1} \to \beta^{-1}$ locally uniformly. It thus follows from (4.17) that $\{u_\varepsilon\}$ is equicontinuous. Then there is a sequence $\varepsilon_n \downarrow 0$ and $u \in BUC(\mathbb{R}^N)$ such that $u_{\varepsilon_n} \to u$ locally uniformly. In view of Proposition IV.1, the existence assertion is proved.

We have in fact shown (4.13) in the process of constructing u. It follows equally well from (4.12) by noting that if u is the solution u of $\beta(u) + H(Du) = n$, then $v(\cdot) = u(\cdot+y)$ is the solution of $\beta(v) + H(Dv) = m$, $m(\cdot) = n(\cdot+y)$. One similarly verifies (4.12) by the construction, however let us observe that it essentially follows from Theorem II.2. Indeed, if $u + H(Du) - n = 0$ and $v + H(Dv) - n = m-n$, Theorem II.2 implies

$$\begin{cases} \gamma_R((u-v)^+) \leq \|(n-m)^+\|_{L^\infty(\mathbb{R}^N)}, \\ R = \max(\|u\|_{L^\infty(\mathbb{R}^n)}, \|v\|_{L^\infty(\mathbb{R}^N)}), \\ \gamma_R(\tau) = \inf\{\beta(s+\tau) - \beta(s) : |s| \leq R\}. \end{cases}$$

which is equivalent to

$$(u-v)^+ \leq \sup\{\beta^{-1}(s+\|(n-m)^+\|_{L^\infty(\mathbb{R}^N)}) - \beta^{-1}(s) : |s| \leq \max(\|m\|_{L^\infty(\mathbb{R}^N)}, \|n\|_{L^\infty(\mathbb{R}^N)})\}.$$

The estimate (4.11) follows from the construction. This ends the sketch of proof.

V. Uniqueness for the Cauchy Problem.

We consider the Cauchy problem for HJ equations. More precisely, we consider the problem

$$(5.1)\qquad \begin{cases} \text{(a)} & u_t + H(x,t,u,Du) = 0 & \text{in} & \Omega \times]0,T] \\ \text{(b)} & u(x,t) = z(x,t) & \text{on} & \partial\Omega \times]0,T] \\ \text{(c)} & u(t,x) = u_0(x) & \text{on} & \Omega. \end{cases}$$

V.1. Viscosity solutions of (5.1).

The notations

$$(5.2)\qquad Q_T = \Omega \times]0,T] \ , \quad Q_T^0 = \Omega \times]0,T[$$

will be used below. The notions of viscosity solutions of (5.1)(a) in Q_T or Q_T^0 is contained in Section II - (in particular, recall Remark 1.13). Let us restate them explicitely for the particular equation (5.1)(a).

Definition 5.1. Let $H \in C(\Omega \times [0,T] \times R \times R^N)$. Then a viscosity subsolution (respectively, supersolution, solution) of $u_t + H(x,t,u,Du) = 0$ on Q_T^0 is a function $u \in C(Q_T^0)$ such that: $\forall \varphi \in \mathcal{D}(Q_T^0)^+$, $k \in R$

$$(5.3)\qquad \begin{cases} E_+(\varphi(u-k),Q_T^0) \neq \phi \implies \exists (x_0,t_0) \in E_+(\varphi(u-k),Q_T^0) \quad \text{such that} \\ -\dfrac{(u(x_0,t_0)-k)}{\varphi(x_0,t_0)}\varphi_t(x_0,t_0) + H(x_0,t_0,u(x_0,t_0), -\dfrac{(u(x_0,t_0)-k)}{\varphi(x_0,t_0)} Du(x_0t_0)) \leq 0 \end{cases}$$

(respectively,

$$(5.4)\qquad \begin{cases} E_-(\varphi(u-k),Q_T^0) \neq \phi \implies \exists (x_0,t_0) \in E_-(\varphi(u-k),Q_T^0) \quad \text{such that} \\ -\dfrac{(u(x_0,t_0)-k)}{\varphi(x_0,t_0)}\varphi_t(x_0,t_0) + H(x_0,t_0,u(x_0,t_0), -\dfrac{(u(x_0,t_0)-k)}{\varphi(x_0,t_0)} Du(x_0,t_0)) \geq 0 \ ; \end{cases}$$

respectively (5.3) and (5.4).)

One defines viscosity subsolutions, etc., in Q_T by replacing Q_T^0 by Q_T everywhere above. A viscosity subsolution (etc.) of (5.1) is a $u \in C(\bar{Q}_T)$ which is a viscosity solution of (5.1)(a) in Q_T such that $u \leq z$ on $\partial\Omega \times [0,T]$, $u(x,0) \leq u_0(x)$ in Ω (etc.).

Owing to the special form of the equation (5.1)(a) with respect to the domain Ω_T we have:

<u>Proposition V.I.</u> Let $u \in C(\Omega_T)$ be a viscosity subsolution (respectively, supersolution, solution) of (5.1)(a) in Ω_T^0. Then u is a viscosity subsolution (respectively, supersolution, solution) of (5.1)(a) in Ω_T.

<u>Proof.</u> It suffices to treat the subsolution case. Let $\varphi \in \mathcal{D}(\Omega_T)^+$, $k \in \mathbb{R}$, u be a viscosity subsolution in Ω_T^0 and $(x_0,t_0) \in E_+(\varphi(u-k),\Omega_T)$. If $0 < t_0 < T$ we choose $\chi \in \mathcal{D}((0,T))$ such that $0 \leq \chi \leq 1$ and $\chi(t_0) = 1$. Then $\chi\varphi \in \mathcal{D}(\Omega_T^0)^+$ and $(x_0,t_0) \in E_+(\chi\varphi(u-k),\Omega_T^0)$. By Theorem I.3 and $\chi'(t_0) = 0$, the inequality of (5.3) holds. If $t_0 = T$ we choose $\chi_\varepsilon \in C^\infty([0,T])$ so that $0 \leq \chi_\varepsilon \leq 1$, $\chi_\varepsilon \equiv 1$ on $[0,T-2\varepsilon]$, $\chi_\varepsilon \equiv 0$ on $[T-\varepsilon,T]$ and $\chi_\varepsilon' \leq 0$. Again $\chi_\varepsilon\varphi \in \mathcal{D}(\Omega_T^0)^+$ Moreover, $\varphi(u-k) > 0$ at (x_0,T) implies $\chi_\varepsilon\varphi(u-k)$ has a positive value for ε small. Let $(x_\varepsilon,t_\varepsilon) \in E_+(\chi_\varepsilon\varphi(u-k),\Omega_T^0)$. Passing to a subsequence if necessary, we assume $(x_\varepsilon,t_\varepsilon) \to (\bar{x},\bar{t}) \in E_+(\varphi(u-k),\Omega_T)$. Then, by Theorem I.3,

$$-\frac{(u(x_\varepsilon,t_\varepsilon)-k)}{\varphi(x_\varepsilon,t_\varepsilon)}\varphi_t(x_\varepsilon,t_\varepsilon) - \frac{(u(x_\varepsilon,t_\varepsilon)-k)}{\chi(t_\varepsilon)}\chi_\varepsilon'(t_\varepsilon)$$

$$+ H(x_\varepsilon,t_\varepsilon,u(x_\varepsilon,t_\varepsilon), -\frac{(u(x_\varepsilon,t_\varepsilon)-k}{\varphi(x_\varepsilon,t_\varepsilon)}D\varphi(x_\varepsilon,t_\varepsilon)) \leq 0 .$$

Now $-(u(x_\varepsilon,t_\varepsilon)-k)\,\chi_\varepsilon'(t_\varepsilon) \geq 0$ so we deduce the inequality of (5.3) with $(\bar{x},\bar{t})$ in place of (x_0,t_0) in the limit. This completes the proof.

<u>Remark 5.5.</u> In the general context of Section I, if $\mathcal{O} \subset \mathcal{O}_1 \cap \mathcal{O} \cup \partial\mathcal{O}$ we roughly have that if $u \in C(\mathcal{O}_1)$ is a viscosity subsolution of $F = 0$ in $\mathcal{O}$ and $F(y,r,p+\lambda\nu(y))$ is nondecreasing in λ for $y \in \mathcal{O}_1\backslash\mathcal{O}$ and $\nu(y)$ the exterior normal to $\mathcal{O}$ at y, then u is a viscosity subsolution in $\mathcal{O}_1$. However, we will not make the assumptions precise.

We will freely use the assertions of Section I concerning viscosity subsolutions, etc., in Ω_T^0 and Ω_T. In this connection we again recall Remark (1.13) as well as the fact that if $u \in C^1(\Omega_T^0)$ and u and Du extend continuously to all of Ω_T, then $u \in C^1(\Omega_T)$, etc.

V.2 Uniqueness of Solutions of the Cauchy Problem.

We first formulate the various assumptions we will use in what follows:

$$(5.6)\quad \begin{cases} H \in C(\bar{\Omega} \times [0,T] \times R \times R^N) \text{ is uniformly continuous in} \\ \bar{\Omega} \times [0,T] \times [-R,R] \times B(0,R) \text{ for each } R > 0 . \end{cases}$$

$$(5.7)\quad \begin{cases} \text{For } R > 0 \text{ there is a } \gamma_R \in R \text{ such that} \\ H(x,t,r,p) - H(x,t,s,p) \geq \gamma_R(r-s) \text{ for } x \in \Omega,\ -R \leq s \leq r \leq R, \\ 0 \leq t \leq T \text{ and } p \in R^n. \end{cases}$$

$$(5.8)\quad \begin{cases} \lim\limits_{\alpha \downarrow 0} \sup\{|H(x,t,s,p)-H(y,t,s,p)| : |x-y|(1+|p|) \leq \alpha,\ 0 \leq t \leq T,\ |s| \leq R\} = 0 \\ \text{for any } R > 0 . \end{cases}$$

$$(5.8^*)\quad \begin{cases} \lim\limits_{\alpha \downarrow 0} \sup\{|H(x,t,s,p)-H(y,t,s,p)| : |x-y| \leq \alpha,\ |x-y||p| \leq R,\ 0 \leq t \leq T,\ |s| \leq R\} = 0 \\ \text{for any } R > 0 . \end{cases}$$

These conditions are obvious analogues of (2.18)-(2.20*). See section V.4 concerning their necessity.

The main uniqueness result is:

<u>Theorem V.2.</u> Let (5.6) and (5.7) hold. Let $u \in C_b(\bar{Q}_T)$ be a viscosity subsolution of $u_t + H(x,t,u,Du) = 0$ in Q_T and $v \in C_b(\bar{Q}_T)$ be a viscosity supersolution of $v_t + H(x,t,v,Dv) = g(x,t)$ in Q_T where $g \in C_b(\bar{Q}_T)$. Let $R_0 = \max(\|u\|_{L^\infty(Q_T)}, \|v\|_{L^\infty(Q_T)})$ and $\gamma = \gamma_{R_0}$ as in (5.7). Set $\partial_0 Q_T = \partial\Omega \times [0,T] \quad (\bar{\Omega} \times \{0\})$. Then:

(i) If (5.8*) holds and $u|_{\partial_0 Q_T}$, $v|_{\partial_0 Q_T} \in BUC(\partial_0 Q_T)$ and

$$\lim_{\substack{(x,t)\in Q_T^0 \\ (x,t)\to(x_0,t_0)}} |u(x,t)-u(x_0,t_0)| + |v(x,t)-v(x_0,t_0)| = 0$$

uniformly for $(x_0,t_0) \in \partial_0 Q_T$, then

$$(5.9) \qquad \|e^{\gamma t}(u-v)^+\|_{L^\infty(Q_T)} \le \|e^{\gamma t}(u-v)^+\|_{L^\infty(\partial_0 Q_T)} + \int_0^T e^{\gamma s}\,\|g(\cdot,s)^-\|_{L^\infty(\Omega)}\,ds .$$

(ii) If (5.8) holds and $u,v \in BUC(\bar{Q}_T)$, then (5.9) holds.

(iii) If $Du, Dv \in L^\infty(Q_T)$, then (5.9) holds.

<u>Remark 5.10.</u> Remarks parallel to (2.6), (2.17) are valid here.

Much of the proof of Theorem V.2 consists of straightforward adaptation of the arguments given in earlier sections and we will not repeat these. Instead we treat a simple model case to exhibit the only new features. To this end, assume $\gamma \in R$,

$$(5.11) \qquad H(x,t,u,p) = \gamma u + \bar{H}(p)$$

and

$$(5.12) \qquad \Omega = R^N \text{ and } u(x,t),\ v(x,t) \to 0 \text{ as } |x| \to \infty \text{ uniformly for } 0 \le t \le T.$$

We will write H in place of $\bar{H}$ above. Now choose $\varphi_\alpha(x) = \varphi(x/\alpha)$, $\psi_\alpha(t) = \psi(t/\alpha)$ where $\varphi \in \mathcal{D}(\mathbb{R}^N)^+$, $\psi \in \mathcal{D}([0,T])^+$, $\varphi(0) = 1$, $\psi(0) = 1$, $0 \le \varphi, \psi \le 1$, $\operatorname{supp}\varphi \subset B(0,1)$, $\operatorname{supp}\psi \subset [-1,1]$. (In the case of (x,t) dependence of H we would require $\varphi(x) = 1 - |x|^2$, $\psi(t) = 1 - t^2$ near $x = 0$, $t = 0$.) Set

$$(5.11) \qquad m_0(t) = \max_{x \in \mathbb{R}^N} (u(x,t) - v(x,t)).$$

Finally, let $\eta \in (]0,T[)^+$ and assume

$$(5.12) \qquad E_+(\eta(m_0-k):]0,T[) \ne \emptyset.$$

Now define

$$(5.13) \qquad M_\alpha = \sup_{\substack{x,y \in \mathbb{R}^N \\ 0 \le t,s \le T}} \eta\left(\frac{t+s}{2}\right)\psi_\alpha(t-s)\varphi_\alpha(x-y)(u(x,t)-v(y,s)-k).$$

Clearly $M_\alpha \ge \eta(m_0-k)$ on $[0,T]$ and

$$(5.14) \qquad M_\alpha \to \max_{[0,T]} \eta(m_0-k) \quad \text{as } \alpha \downarrow 0.$$

Let $x_\alpha, y_\alpha \in \mathbb{R}^N$, $t_\alpha, s_\alpha \in [0,T]$ be such that

$$M_\alpha = \eta\left(\frac{t_\alpha+s_\alpha}{\alpha}\right)\psi_\alpha(t_\alpha-s_\alpha)\ {}_\alpha(x_\alpha-y_\alpha)(u(x_\alpha,t_\alpha)-v(y_\alpha,s_\alpha)-k). \tag{5.15}$$

Because $|x_\alpha-y_\alpha| \leq \alpha$ and $u,v \to 0$ at ∞ uniformly ((5.12)), we may assume (using subsequences if necessary) that $x_\alpha, y_\alpha \to x_0, x_0$ and $t_\alpha, s_\alpha \to t_0, t_0$ as $\alpha \downarrow 0$. Moreover, by (5.14),

$$t_0 \in E_+(\eta(m_0-k)) \tag{5.16}$$

and so $t_0 > 0$. Then

$$\eta((\cdot+s_\alpha)/2)\psi_\alpha(\cdot-s_\alpha)\varphi_\alpha(\cdot-y_\alpha) \quad \text{and} \quad \eta((t_\alpha+\cdot)/2)\psi_\alpha(t_\alpha-\cdot)\varphi_\alpha(x_\alpha-\cdot)$$

are in $\mathcal{D}(Q_T)^+$ for α small and using the assumed proprties of u,v we find

$$-\left[\frac{\eta'\left(\frac{t_\alpha+s_\alpha}{2}\right)}{2\eta\left(\frac{t_\alpha+s_\alpha}{2}\right)} + \frac{\psi_\alpha'(t_\alpha-s_\alpha)}{\psi_\alpha(t_\alpha-s_\alpha)}\right](u(x_\alpha,t_\alpha) - v(y_\alpha,s_\alpha) - k)$$

$$+\gamma\, u(x_\alpha,t_\alpha) + H\left(-\frac{(u(x_\alpha,t_\alpha)-v(y_\alpha,s_\alpha) - k)}{\varphi(x_\alpha-y_\alpha)}(D\varphi_\alpha)(x_\alpha-y_\alpha)\right) \leq 0,$$

or

$$-\left[\frac{\eta'\left(\frac{t_\alpha+s_\alpha}{2}\right)}{2\eta\left(\frac{t_\alpha+s_\alpha}{2}\right)} - \frac{\psi_\alpha'(t_\alpha-s_\alpha)}{\psi_\alpha(t_\alpha-s_\alpha)}\right](v(y_\alpha,s_\alpha) - u(x_\alpha,t_\alpha) + k)$$

$$+\gamma v(y_\alpha,s_\alpha) + H\left(-\frac{(u(x_\alpha,t_\alpha) - v(y_\alpha,s_\alpha) - k)}{\varphi_\alpha(x_\alpha-y_\alpha)}(D\varphi_\alpha)(x_\alpha-y_\alpha)\right) \geq g(y_\alpha,s_\alpha).$$

Combining these inequalities we find

$$-\frac{\eta'\left(\frac{t_\alpha+s_\alpha}{2}\right)}{\eta\left(\frac{t_\alpha+s_\alpha}{2}\right)}(u(x_\alpha,t_\alpha)-v(y_\alpha,s_\alpha)-k) + \gamma(u(x_\alpha,t_\alpha)-v(y_\alpha,s_\alpha))$$

$$\leq -g(y_\alpha,s_\alpha) \leq \|g(\cdot,s_\alpha)^-\|_{L^\infty(\mathbb{R}^N)}.$$

Now let $\alpha \downarrow 0$ to find

$$(5.17)\qquad \begin{aligned} &-\frac{\eta'(t_0)}{\eta(t_0)}\,(u(x_0,t_0) - v(x_0,t_0)-k) + \gamma(u(x_0,t_0) - v(x_0,t_0)) \\ &\qquad \le \|g(\cdot,t_0)^-\|_{L^\infty(\mathbb{R}^N)} . \end{aligned}$$

We also claim that $m_0(t_0) = u(x_0,t_0) - v(x_0,t_0)$, which is in fact clear. Let us review the outcome of the above that we need. If m_0 is given by (5.11), $\eta \in (]0,T[)$, and (5.12) holds, we have produced $t_0 \in E_+(\eta(m_0(t)-k))$ such that (5.17) holds, which is

$$-\frac{\eta'(t_0)}{\eta(t_0)}\,(m_0(t_0)-k) + \gamma m_0(t_0) \le \|g(\cdot,t_0)^-\|_{L^\infty(\mathbb{R}^N)} .$$

By Corollary I.12 we conclude,

$$e^{\gamma t} m_0(t) \le m_0(0) + \int_0^t e^{\gamma s}\|g(\cdot,s)^-\|_{L^\infty(\mathbb{R}^N)}\,ds ,$$

which completes the proof.

V.3. The Cone of Dependence

We are out to show that if u,v are two viscosity solutions of

$$(5.18)\qquad u_t + H(x,t,u,Du) = 0 \qquad \text{in } \mathbb{R}^N \times]0,T]$$

with $u(x,0) = v(x,0)$ on some ball $|x| \le R$, then - under natural assumptions - $u = v$ on the cone $|x| \le R - Lt$ where L is a Lipschitz constant for $H(x,t,r,p)$ in p. We assume

$$(5.19)\qquad \begin{cases} H \in C(\mathbb{R}^N \times [0,T] \times \mathbb{R} \times \mathbb{R}^N) \text{ and } H(x,t,r,p) \text{ is nondecreasing in } r \\ \text{for } (x,t,p) \in \mathbb{R}^N \times [0,T] \times \mathbb{R}^N . \end{cases}$$

The main result is:

Theorem V.4. Let $u,v \in C(\mathbb{R}^N \times [0,T])$ be viscosity solutions of (5.18) on $Q_T = \mathbb{R}^N \times [0,T]$. Let (5.19) hold and

$$u(x,0) \le v(x,0) \quad \text{on} \quad |x| \le R, \tag{5.20}$$

$$C = \max(\|Du\|_{L^\infty(Q_T)}, \|Dv\|_{L^\infty(Q_T)}), \quad m = \max(\|u\|_{L^\infty(Q_T)}, \|v\|_{L^\infty(Q_T)}) \tag{5.21}$$

and

$$\begin{cases} |H(x,t,r,p) - H(x,t,r,q)| \le L|p-q| \text{ for } |p|, |q| \le C, |r| \le m, |x| \le R - Lt, \\ \text{and } 0 \le t \le T. \end{cases} \tag{5.22}$$

Then

$$u \le v \quad \text{on} \quad |x| \le R - Lt, \ 0 \le t \le T. \tag{5.23}$$

Moreover, this is correct if $C = \infty$ in (5.22), $u,v \in C(\bar{Q}_T)$, and $H(x,t,r,p)$ is continuous in (x,t) uniformly for $|r| \le m$, $p \in \mathbb{R}^N$.

This result is a consequence of the following proposition:

Proposition V.5. Let (5.19) hold and $u,v \in C(\bar{Q}_T)$ be viscosity solutions of (5.18) on Q_T. Let $\Lambda \in C^1(\bar{Q}_T)$, $\Lambda \ge 0$, $\Lambda = 0$ for $|x|$ large and

$$-\Lambda_t > L|D\Lambda| \text{ in } (\operatorname{supp}\Lambda)^0 \text{ (the interior of } \operatorname{supp}\Lambda). \tag{5.24}$$

Assume (5.21) and that (5.22) holds for $(x,t) \in (\operatorname{supp}\Lambda)^0$. If $u(x,0) \le v(x,0)$ on $\{(x,0): \Lambda(x,0) > 0\}$, then $u \le v$ on $\operatorname{supp}\Lambda$. Moreover, the result is valid if $C = \infty$ in (5.22), $u,v \in C(\bar{Q}_T)$ and $H(x,t,r,p)$ is continuous in (x,t) uniformly for $|r| \le m$, $p \in \mathbb{R}^N$.

We prove the Theorem from the Proposition and then prove the Proposition.

Proof of Theorem V.4. Consider

$$\Lambda(x,t) = g(R_0 - Lt - \lambda|x|^{1+\alpha})$$

where $g \in C^\infty(\mathbb{R})$, $g(r) = 0$ if $r \le 0$, $g'(r) > 0$ if $r > 0$. One has

$$\operatorname{supp}\Lambda = \{(x,t): 0 \le t \le R/L, \ |x| \le (\lambda^{-1}(R_0 - Lt))^{\frac{1}{1+\alpha}}\}$$

so

$\{x:\Lambda(x,0) > 0\} = \{|x| < (\lambda^{-1}R_0)^{\frac{1}{1+\alpha}}\}$. We choose λ,α so that

$$(5.25) \qquad (\lambda^{-1}R_0)^{\frac{1}{1+\alpha}} \leq R \quad \text{or} \quad 1 \leq \lambda R^{1+\alpha}/R_0$$

whence (5.20) implies $u(x,0) \leq v(x,0)$ on $\operatorname{supp}\Lambda(\cdot,0)$. Now $g'(R_0 - Lt - \lambda|x|^{1+\alpha}) > 0$ on $(\operatorname{supp}\Lambda)^0$ and

$$L|D\Lambda| = L\lambda(1+\alpha)|x|^{\alpha}g'(R_0 - Lt - \lambda|x|^{1+\alpha})$$

$$-\Lambda_t = L\, g'(R_0 - Lt - \lambda|x|^{1+\alpha}).$$

If

$$(5.26) \qquad \lambda(1+\alpha)R_0^{\alpha} < 1$$

we have $-\Lambda_t > L|D\Lambda|$ on $(\operatorname{supp}\Lambda)^0$. The proposition implies $u \leq v$ on $\operatorname{supp}\Lambda$. We will be done once we show that we can choose $\lambda = \lambda(\alpha)$, $R_0 = R_0(\alpha)$ so that (5.25), (5.26) hold and $\lambda(\alpha) \to 1$, $R_0(\alpha) \to R$ as $\alpha \downarrow 0$. Put $(1+2\alpha)R_0^{1+\alpha} = R^{1+\alpha}$. Then (5.26), (5.25) become

$$\frac{1}{(1+2\alpha)^{1/(1+\alpha)}R^{\alpha}} \leq \lambda < \frac{(1+2\alpha)^{\frac{\alpha}{(1+\alpha)}}}{(1+\alpha)\,R^{\alpha}}$$

so we may use $\lambda(\alpha) = 1/((1+2\alpha)^{\frac{1}{1+\alpha}}R^{\alpha})$. The proof is complete.

<u>Proof of Proposition V.5.</u> Let $\varphi_\alpha,\psi_\alpha$ be as in the proof of Theorem V.2 and u,v,Λ as in the Proposition. We assume

$$M = \max_{\substack{\operatorname{supp}\Lambda \\ 0\leq t\leq T}} \Lambda^2(u-v) > 0$$

and will reach a contradiction.

Set

$$M_\alpha = \max_{Q_T\times Q_T} \varphi_\alpha(x-y)\psi_\alpha(t-s)\Lambda(x,t)\Lambda(y,s)(u(x,t)-v(y,s))$$

$$= \varphi_\alpha(x_\alpha-y_\alpha)\psi_\alpha(t_\alpha-s_\alpha)\Lambda(x_\alpha,t_\alpha)\Lambda(y_\alpha,s_\alpha)(u(x_\alpha,t_\alpha)-v(y_\alpha,s_\alpha)).$$

Clearly $M_\alpha \to M > 0$ and so $t_\alpha, s_\alpha > \delta > 0$ and (x_α,t_α), $(y_\alpha,s_\alpha) \in \operatorname{supp}\Lambda^0$ for α small. Thus

$$\begin{cases} -\dfrac{\psi'_\alpha}{\psi_\alpha}(u-v) - \dfrac{\partial\Lambda}{\partial t}\dfrac{(u-v)}{\Lambda} + H\left(x_\alpha,t_\alpha,u,(u-v)\left(\dfrac{D_x\varphi_\alpha}{\varphi_\alpha} + \dfrac{D_x\Lambda}{\Lambda}\right)\right) \le 0 \\[2ex] +\dfrac{\psi'_\alpha}{\psi_\alpha}(v-u) - \dfrac{\partial\Lambda}{\partial s}\dfrac{(u-v)}{\Lambda} + H\left(y_\alpha,s_\alpha,v,-(u-v)\left(\dfrac{D_x\varphi_\alpha}{\varphi_\alpha} - \dfrac{D_y\Lambda}{\Lambda}\right)\right) \ge 0 \end{cases}$$

where the reader can keep track of the correct arguments in each term. Subtracting these yields

$$-\frac{\partial\Lambda}{\partial t}(x_\alpha,t_\alpha)\frac{u-v}{\Lambda(x_\alpha,t_\alpha)} - \frac{\partial\Lambda}{\partial t}(y_\alpha,s_\alpha)\frac{u-v}{\Lambda(x_\alpha,t_\alpha)} +$$

$$H\left(x_\alpha,t_\alpha,u,(u-v)\left(\frac{D_x\varphi_\alpha}{\varphi_\alpha} + \frac{D_x\Lambda}{\Lambda}\right)\right) - H\left(y_\alpha,s_\alpha,v,-(u-v)\left(\frac{D_x\varphi_\alpha}{\varphi_\alpha} - \frac{D_y\Lambda}{\Lambda}\right)\right) \le 0 .$$

Since $u(x_\alpha,t_\alpha) \ge v(y_\alpha,s_\alpha)$, (5.19) allows us to replace v by u in the third argument of H above. Now, since (x_α,t_α), $(y_\alpha,s_\alpha) \to (x_0,t_0) \in (\operatorname{supp}\Lambda)^0$ and

$$\left|(u(x_\alpha,t_\alpha) - v(y_\alpha,s_\alpha))\left(\frac{(D\varphi_\alpha)(x_\alpha-y_\alpha)}{\varphi_\alpha(x_\alpha-y_\alpha)} + \frac{D\Lambda(x_\alpha,t_\alpha)}{\Lambda(x_\alpha,t_\alpha)}\right)\right| \le C$$

$$\left|(u(x_\alpha,t_\alpha)-v(y_\alpha-s_\alpha))\left(\frac{(D\varphi_\alpha)(x_\alpha-y_\alpha)}{\varphi_\alpha(x_\alpha-y_\alpha)} - \frac{D\Lambda(y_\alpha,s_\alpha)}{\Lambda(y_\alpha,s_\alpha)}\right)\right| \le C$$

by (5.21) and Lemma II.3 we may let $\alpha \downarrow 0$ above and use (5.22) to conclude

$$-2\frac{\partial\Lambda}{\partial t}(x_0,t_0)(u-v)(x_0,t_0) - 2L|D\Lambda(x_0,t_0)|(u-v)(x_0,t_0) \le 0$$

which contradicts $-\Lambda_t > L|D\Lambda|$ on $(\operatorname{supp}\Lambda)^0$. This passage to the limit is valid if $C < \infty$. If $C = \infty$ it is valid under the assumption that $H(x,t,r,p)$ is uniformly continuous in (x,t) for $|r| \le m$, $p \in R^N$.

<u>Remark 5.27.</u> There are many possible variants of these results, including continuous dependence of solutions of $u_t + H(x,t,u,Du) = g$ in the cone of dependence on $u(x,0)$ in $|x| \le R$ and g in $|x| \le R-Lt$. But it is obvious how to obtain these.

<u>Remark 5.28.</u> Results in the spirit of Theorem V.4 are given in A. Friedman [16], S. N. Kruzkov [20] and P. L. Lions [22]. However, these all deal with generalized $(W^{1,\infty})$ solutions obtained via the vanishing viscosity method rather than intrinsically characterized solutions.

Remark 5.29. The assumption $C < \infty$ in (5.21) is a stringent requirement - but certainly a necessary one in general. Typical existence theorems provide $W^{1,\infty}$ solutions in any case (e.g. [13], [16], [22]).

V.4. Examples of Nonuniqueness.

Let $b \in C(\mathbb{R})$. If the solutions of

$$(5.30) \qquad \begin{cases} \dfrac{dx}{dt} = b(x) , \\ x(0) = x_0 \end{cases}$$

are "too" nonunique, then bounded viscosity solutions of

$$(5.31) \qquad \begin{cases} u_t + b(x)u_x = 0 , \quad t > 0 , \quad x \in \mathbb{R}, \\ u(x,0) = u_0(x), \end{cases}$$

will also not be unique.

Let us make this precise. Assume for every $x_0 \in \mathbb{R}$ we may choose a solution $x = X(t,x_0)$ of (5.30) defined for $t \in \mathbb{R}$ in such a way that: $X(t,x_0)$ is continuous in (t,x_0), $x_0 \to X(t,x_0)$ is a homeomorphism of $\mathbb{R}$ for each $t \in \mathbb{R}$ and $X(t,X(\tau,x_0)) = X(t+\tau,x_0)$ for $t,\tau,x_0 \in \mathbb{R}$ (i.e., X is a "flow" or one parameter group). We claim that then

$$(5.32) \qquad u(x,t) \equiv u_0(X(-t,x))$$

is a viscosity solution of (5.31). The initial condition is clearly satisfied. Let $\varphi \in \mathcal{D}(\mathbb{R} \times (0,\infty))^+$, $k \in \mathbb{R}$ and $(\bar{x},\bar{t}) \in E_+(\varphi(u-k))$. Then, by (5.32),

$$\varphi(\bar{x},\bar{t})(u(\bar{x},\bar{t})-k) = \varphi(\bar{x},\bar{t})(u_0(X(-\bar{t},\bar{x}))-k) \geq \varphi(x,t)(u_0(X(-t,x))-k)$$

for all t and x. Put $x = X(t-\bar{t},\bar{x})$ in this inequality to find

$$\varphi(\bar{x},\bar{t})(u(\bar{x},\bar{t})-k) \geq \varphi(X(t-\bar{t},\bar{x}),t)(u(\bar{x},\bar{t})-k)$$

for all t. This implies that $t \to \varphi(X(t-\bar{t},\bar{x}),t)$ is maximized at $t = \bar{t}$ and so

$$\frac{d}{dt}\varphi(X(t-\bar{t},\bar{x}),t)\Big|_{t=\bar{t}} = \varphi_t(\bar{x},\bar{t}) + b(\bar{x})\varphi_x(\bar{x},\bar{t}) = 0 .$$

Multiplying this relation by $(u(\bar{x},\bar{t})-k)/\varphi(\bar{x},\bar{t})$ we find u is a viscosity subsolution of $u_t + bu_x = 0$. Similarly, it is a supersolution and so a solution.

Nonuniqueness arises when X may be chosen in more than one way. In [3] examples of this may be found. The simplest have the following structure: There are classes F of continuously differentiable homeomorphisms of R such that for $f,g \in F$ one has $f'(f^{-1}(x)) \equiv g'(g^{-1}(x))$. If $f \neq g$ and $b(x) = f'(f^{-1}(x))$, then

$$X_1(t,x_0) = f(t + f^{-1}(x_0)), \quad X_2(t,x_0) = g(t + g^{-1}(x_0))$$

are distinct flows with the desired properties. More complex examples in higher dimensions are also given in [3].

While this example is for the pure Cauchy problem, it may be regarded as a Dirichlet problem in a half space. To get the Hamiltonian to be increasing in the unknown, set $v = e^{-\gamma t}u$ in (5.31) so that it becomes

$$\begin{cases} v_t + \gamma v + b(x)v_x = 0 \\ v(x,0) = u_0(x). \end{cases}$$

VI. Existence of Viscosity Solutions for the Cauchy Problem.

As in Section IV, we will restrict ourselves to a few remarks. Two of the basic ways to produce solutions of the Cauchy problem are the vanishing viscosity method and numerical approximation. If the method of vanishing viscosity converges, the result will be a viscosity solution (Theorem VI.1). This fact may be used in a straightforward way to obtain many new existence and uniqueness theorems. This is indicated by the very general results stated for the simple model problem of Paragraph IV.2. The relationship to the nonlinear semigroup theory is touched on in Section VI.3. Convergence of numerical schemes to viscosity solutions is discussed in [8].

IV.1 Vanishing Viscosity and Viscosity Solutions.

Avoiding useless repetition, we rely on the reader to adapt the proof of Proposition IV.1 and establish:

Proposition VI.1. Let u_ε be a solution of

$$(6.1)\qquad \begin{cases} u_{\varepsilon t} - \varepsilon\Delta u_\varepsilon + H_\varepsilon(x,t,u_\varepsilon,Du_\varepsilon) = 0 & \text{in } Q_T, \\ u_\varepsilon = z_\varepsilon \text{ on } \partial\Omega \times [0,T],\ u_\varepsilon(x,0) = u_{0\varepsilon}(x) & \text{in } \bar{\Omega}, \end{cases}$$

with $u_{\varepsilon t}, u_{\varepsilon x_i x_j} \in C(Q_T)$ and $u \in C_b(\bar{Q}_T)$. Assume $H_\varepsilon \to H$ in $C(Q_T \times R \times R^N)$, $z_\varepsilon \to z$ in $C(\partial\Omega \times [0,T])$ and $u_{0\varepsilon} \to u_0$ in $C(\bar{\Omega})$. If $\varepsilon_n \downarrow 0$ and $u_{\varepsilon_n} \to u$ in $C(Q_T)$, then u is a viscosity solution of

$$(6.2)\qquad u_t + H(x,t,u,Du) = 0 \quad \text{in} \quad Q_T .$$

If the convergence $u_{\varepsilon_n} \to u$ is in $C(\bar{Q}_T)$, then u also satisfies

$$(6.3)\qquad u = z \text{ on } \partial\Omega \times [0,T],\ u(x,0) = u_0(x) \text{ in } \bar{\Omega} .$$

VI.2. A Model Problem.

Let

$$H \in C(\mathbb{R}^N)\ ,\quad u_0 \in BUC(\mathbb{R}^N) \tag{6.4}$$

and consider the problem

$$\begin{cases} \text{(i)} & u_t + H(Du) = 0 \quad \text{in} \quad \mathbb{R}^N \times]0,\infty[= Q \\ \text{(ii)} & u(x,0) = u_0(x) \quad \text{in} \quad \mathbb{R}^N . \end{cases} \tag{6.5}$$

Our main existence result for (6.5) is:

Theorem VI.1. Let (6.4) hold. Then there is a unique $u \in C(\bar{Q}) \cap C_b(\bar{Q}_T)$ for all $T > 0$ which is a viscosity solution of $u_t + H(Du) = 0$ and Q and satisfies

$$\lim_{t\downarrow 0} \|u(\cdot,t) - u_0(\cdot)\|_{L^\infty(\mathbb{R}^N)} = 0 . \tag{6.6}$$

Moreover,

$$|u(x,t)-u(y,t)| \le \sup_{\xi\in\mathbb{R}^N} |u_0(\xi) - u_0(\xi+y-x)| \quad \text{for} \quad x,y \in \mathbb{R}^N,\ t \ge 0 . \tag{6.7}$$

Finally, if $S(t):BUC(\mathbb{R}^N) \to BUC(\mathbb{R}^N)$ is defined for $t \ge 0$ by $S(t)u_0 = u(\cdot,t)$, then S is a strongly continuous nonexpansive semigroup on $BUC(\mathbb{R}^N)$ such that

$$\|(S(t)u_0 - S(t)v_0)^+\|_{L^\infty(\mathbb{R}^N)} \le \|(u_0-v_0)^+\|_{L^\infty(\mathbb{R}^N)} \quad \text{for} \quad u_0 v_0 \in BUC(\mathbb{R}^N) . \tag{6.8}$$

The existence of u satisfying (6.6), (6.7) is easily established by the vanishing viscosity method, and we will not carry this out. (The proof of Theorem IV.3 indicates the main points.) The uniqueness and the estimate (6.8) follow from Theorem V.2. The uniqueness implies the semigroup property $S(t)S(\tau) = S(t+\tau)$ for $t,\tau \ge 0$ as usual. We remark that (6.7) also follows from (6.8) and the translation invariance of this model problem as reflected in

$$v_0(x+y) = u_0(x) \implies (S(t)v_0)(x+y) = (S(t)u_0)(x).$$

Actually, Theorem IV.1 follows directly from Theorem IV.3 and nonlinear semigroup theory, as recalled next.

VI.3. An m-Accretive Operator.

Several authors, in particular Aizawa [1] and Tamburro [25], recognized that nonlinear semigroup theory provides solutions to the Cauchy problem for HJ equations. We just sketch this here in our new context for our model problem.

Let $H \in C(\mathbb{R}^N)$. Define an operator A in $BUC(\mathbb{R}^N)$ by $u \in BUC(\mathbb{R}^N)$ is in $D(A)$ if there is an $g \in BUC(\mathbb{R}^N)$ for which $H(Du) = g$ in the viscosity sense and then set $Au = g$. It follows from Proposition IV.3 that for each $m \in BUC(\mathbb{R}^N)$ and $\lambda > 0$ the problem $u + \lambda Au = m$ has a unique viscosity solution $u \in D(A)$. Denote this solution by $u = J_\lambda m$, $J_\lambda = (I + \lambda A)^{-1}$. It also follows from Proposition IV.3 that

$$(6.9)\qquad \begin{aligned} &\text{(i)} \quad \|(J_\lambda m - J_\lambda n)^+\|_{L^\infty(\mathbb{R}^N)} \le \|(m-n)^+\|_{L^\infty(R^N)}, \\ &\text{(ii)} \quad \|(J_\lambda m - J_\lambda n)\|_{L^\infty(\mathbb{R}^N)} \le \|(m-n)\|_{L^\infty(R^N)}, \end{aligned}$$

for $m,n \in BUC(\mathbb{R}^N)$. The condition (4.9)(ii) is the definition of "A is accretive" in $BUC(\mathbb{R}^N)$. The fact that also $R(I+\lambda A) = BUC(\mathbb{R}^N)$ is by definition "A is m-accretive" in $BUC(\mathbb{R}^N)$. Clearly $D(A)$ is dense in $BUC(\mathbb{R}^N)$. By the Crandall-Liggett Theorem (see, e.g., [2],[7],[11]), the functions $u_\varepsilon:[0,\infty] \to BUC(\mathbb{R}^N)$ defined for $\varepsilon > 0$ by

$$(6.10)\qquad \begin{cases} \text{(i)} & u_\varepsilon(0) = u_0 \\ \text{(ii)} & \dfrac{u_\varepsilon(t+\varepsilon)-u_\varepsilon(t)}{\varepsilon} + Au_\varepsilon(t+\varepsilon) = 0 \quad \text{for} \quad t > 0 \end{cases}$$

converge in $BUC(\mathbb{R}^N)$ uniformly on compact t-sets as $\varepsilon \downarrow 0$ to a limit

$$\lim_{\varepsilon\downarrow 0} u_\varepsilon(t) = \lim_{\varepsilon\downarrow 0} (I+\varepsilon A)^{-[t/\varepsilon]}u_0 = S(t)u_0$$

where $S(t)$ is a strongly continuous nonexpansive semigroup on $BUC(\mathbb{R}^N)$. We claim $S(t)u_0$ is the viscosity solution of (6.5). Indeed, let $u = S(t)u_0$, $k \in \mathbb{R}$, $\varphi \in \mathcal{D}(Q)^+$ and $E_+(\varphi(u-k)) \neq 0$. Set $u_\varepsilon(x,t) = u_\varepsilon(t)(x)$. Since $u_\varepsilon \to u$ locally uniformly, $E_+(\varphi(u_\varepsilon-k)) \neq \phi$ for ε sufficiently small. (The discontinuities of $u_\varepsilon(x,t)$ at

$t = j\varepsilon$ are no problem.) Moreover, if $(x_\varepsilon, t_\varepsilon) \in E_+(\varphi(u_\varepsilon - k))$ we may assume $(x_\varepsilon, t_\varepsilon) \to (x_0, t_0) \in E_+(\varphi(u-k))$. (*Pass to a subsequence if necessary or choose* φ *so that* $E_+(\varphi(u-k)) = \{(x_0, t_0)\}$.) We have

$$\varphi(x_\varepsilon, t_\varepsilon)(u_\varepsilon(x_\varepsilon, t_\varepsilon) - k) \geq \varphi(x,t)(u_\varepsilon(x,t) - k). \tag{6.11}$$

Since $x_\varepsilon \in E_+(\varphi(\cdot, t_\varepsilon)(u_\varepsilon(\cdot, t_\varepsilon) - k), R^N)$, the definition of A and (4.10)(ii) yield

$$\frac{u_\varepsilon(x_\varepsilon, t_\varepsilon) - u_\varepsilon(x_\varepsilon, t_\varepsilon - \varepsilon)}{\varepsilon} + H\Big(\frac{-(u_\varepsilon(x_\varepsilon, t_\varepsilon) - k)}{\varphi(x_\varepsilon, t_\varepsilon)} D\varphi(x_\varepsilon, t_\varepsilon)\Big) \leq 0 . \tag{6.12}$$

Now, by (6.11)

$$\left\{\begin{aligned} &\varphi(x_\varepsilon, t_\varepsilon)(u_\varepsilon(x_\varepsilon, t_\varepsilon) - u_\varepsilon(x_\varepsilon, t_\varepsilon - \varepsilon)) = \\ &\varphi(x_\varepsilon, t_\varepsilon)(u_\varepsilon(x_\varepsilon, t_\varepsilon) - k) - \varphi(x_\varepsilon, t_\varepsilon - \varepsilon)(u_\varepsilon(x_\varepsilon, t_\varepsilon - \varepsilon) - k) \\ &-(\varphi(x_\varepsilon, t_\varepsilon) - \varphi(x_\varepsilon, t_\varepsilon - \varepsilon))(u_\varepsilon(x_\varepsilon, t_\varepsilon - \varepsilon) - k) \\ &\geq -(\varphi(x_\varepsilon, t_\varepsilon) - \varphi(x_\varepsilon, t_\varepsilon - \varepsilon))(u_\varepsilon(x_\varepsilon, t_\varepsilon - \varepsilon) - k). \end{aligned}\right. \tag{6.13}$$

Using (6.13) in (6.12) yields

$$-\frac{1}{\varphi(x_\varepsilon, t_\varepsilon)} \frac{(\varphi(x_\varepsilon, t_\varepsilon) - (x_\varepsilon, t_\varepsilon - \varepsilon))}{\varepsilon} (u_\varepsilon(x_\varepsilon, t_\varepsilon - \varepsilon) - k)$$

$$+ H\Big(-\frac{(u_\varepsilon(x_\varepsilon, t_\varepsilon) - k)}{\varphi(x_\varepsilon, t_\varepsilon)} D\varphi(x_\varepsilon, t_\varepsilon)\Big) \leq 0 .$$

Letting $\varepsilon \downarrow 0$ we find

$$-\frac{(u(x_0, t_0) - k)}{\varphi(x_0, t_0)} \varphi_t(x_0, t_0) + H\Big(-\frac{(u(x_0, t_0) - k)}{\varphi(x_0, t_0)} D\varphi(x_0, t_0)\Big) \leq 0$$

and u is a viscosity subsolution. Similiarly, it is a supersolution and the claim is proved.

We make some further remarks below which help to clarify the relationship between the notions of viscosity solutions and accretivity. (Only the reader who is familiar with accretivity in spaces of continuous functions and its characterization via duality we see the remarks in this light.) Assume $H \in C(\Omega \times R^N)$, $g \in C(\Omega)$ and $u \in C_b(\Omega)$. If u is a viscosity solution of

$$H(x,Du) \leq g(x) \quad \text{in} \quad \Omega \tag{6.14}$$

and $\psi \in C^1(\Omega)$, then the results of Section I imply

$$H(x,D\psi(x)) \leq g(x) \quad \text{on} \quad E_+(u-\psi). \tag{6.15}$$

The converse also holds. To see this, recall (Proposition I.19) that (6.14) is equivalent to

$$H(x,a) \leq g(x) \quad \text{for} \quad x \in \Omega \quad \text{and} \quad a \in D^+u(x). \tag{6.16}$$

Moreover, $a \in D^+(u(x))$ is equivalent (since u is bounded) to the existence of $\psi \in C^1(\Omega) \cap C_b(\Omega)$ such that $u(x) = \psi(x)$, $D\psi(x) = a$ and $\psi > u$ on $\Omega \setminus \{x\}$. Then $\{x\} = E_+(u - (\psi-1))$ and from (6.15) (applied to $\psi-1$ in place of ψ) we deduce (6.16).

Arguing similarly with supersolutions one concluded that $H(x,Du) = g$ in the viscosity sense if and only if $(g - H(x,D\psi))(u-\psi) \geq 0$ on $E_+(u-\psi) \cup E_-(u-\psi)$ for every $\psi \in C_1(\Omega) \cap C_b(\Omega)$.

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Problems involving Hamilton-Jacobi equations which we take to be either of the stationary form $H(x,u,Du) = 0$ or of the evolution form $u_t + H(x,t,u,Du) = 0$, where Du is the spatial gradient of u - arise in many contexts. Classical analysis of associated problems under boundary and/or initial conditions by the method of characteristics is limited to local considerations owing to the crossing of characteristics. Global analysis of these problems has been hindered by the lack of an appropriate notion of solution for which one has the desired existence and uniqueness

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20. Abstract (continued)

properties. In this work a notion of solution is proposed which allows, for example, solutions to be nowhere differentiable but for which strong uniqueness theorems, stability theorems and general existence theorems, as discussed herein, are all valid.

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